Optimal Policy for a Stochastic Scheduling Problem with an Application to Surgical Scheduling

Abstract

We consider the stochastic, single-machine earliness/tardiness problem (SET), with the sequence of processing of the jobs and their due-dates as decisions and the objective of minimizing the sum of the expected earliness and tardiness costs over all the jobs. In a recent paper, Baker (2014) shows the optimality of the Shortest-Variance-First (SVF) rule under the following two assumptions: (a) The processing duration of each job follows a normal distribution. (b) The earliness and tardiness cost parameters are the same for all the jobs. In this paper, we consider problem SET under assumption (b). We generalize Baker's result by establishing the optimality of the SVF rule for more general distributions of the processing durations and a more general objective function. Specifically, we show that the SVF rule is optimal under the assumption of *dilation ordering* of the processing durations. Since *convex ordering* implies dilation ordering (under finite means), the SVF sequence is also optimal under convex ordering of the processing durations. We also study the effect of variability of the processing durations of the jobs on the optimal cost. An application of problem SET in surgical scheduling is discussed.

1 Introduction

The single-machine earliness/tardiness problem (SET) with random processing times of the jobs is one of the fundamental problems in stochastic scheduling theory; see e.g., Cheng (1991), Soroush (1999), Xia et. al. (2008), and Baker (2014). This problem is defined as follows: Consider n jobs, indexed by $j = 1, 2, \dots n$, that are all available to be processed on a single machine at time 0. The processing duration p_j of job j is a random variable with a known distribution. The processing durations are assumed to be independent of each other and job pre-emption is not allowed. We denote the current time by 0 (i.e., the earliest time a job can start processing is 0). We are to decide the *sequence* in which the jobs are to be processed and also their *due-dates*. Let $\boldsymbol{\pi} = (\pi_1, \pi_2, \dots, \pi_n)$ denote an arbitrary sequence in which the jobs are processed; thus, π_j denotes the j^{th} job in the sequence $\boldsymbol{\pi}$. Let $\mathbf{p}^\pi = (p_{\pi_1}, p_{\pi_2}, \dots, p_{\pi_n})$. Let d_j^π denote the due-date of job π_j in the sequence $\boldsymbol{\pi}$ and let $\mathbf{d}^\pi = (d_1^\pi, d_2^\pi, \dots, d_n^\pi)$. The realized completion time of job π_j in the sequence $\boldsymbol{\pi}$ is $C_j^\pi = \sum_{i=1}^{j} p_{\pi_i}$. The earliness and tardiness of job

π_j in the sequence $\boldsymbol{\pi}$, denoted by E_j^π and T_j^π respectively, are defined as follows: $E_j^\pi = (d_j^\pi - C_j^\pi)^+$ and $T_j^\pi = (C_j^\pi - d_j^\pi)^+$, where $x^+ = \max\{0, x\}$. Let the unit earliness and tardiness costs for job j be α_j and β_j, respectively. The objective is to minimize the sum of the total expected earliness cost and the total expected tardiness cost, with the sequence $\boldsymbol{\pi}$ and due-dates $\mathbf{d}^\pi$ as decisions. Let $F(\boldsymbol{\pi}, \mathbf{d}^\pi)$ denote the expected cost corresponding to $(\boldsymbol{\pi}, \mathbf{d}^\pi)$. Formally, problem SET is defined as follows:

$$\min_{\boldsymbol{\pi}, \mathbf{d}^\pi} \quad F(\boldsymbol{\pi}, \mathbf{d}^\pi) \quad = \quad \min_{\boldsymbol{\pi}, \mathbf{d}^\pi} \quad \sum_{j=1}^{n} \mathbb{E}[\alpha_{\pi_j} E_j^\pi + \beta_{\pi_j} T_j^\pi]. \tag{SET}$$

In a recent paper, Baker (2014) studied Problem SET under the assumption that the processing duration of job j, $j = 1, 2, \ldots, n$, follows a normal distribution with mean μ_j and variance σ_j^2, i.e., $p_j \sim N(\mu_j, \sigma_j^2)$, and developed a Branch-and-Bound algorithm to find an optimal solution and reported its computational performance. He also studied the special case in which all the jobs have common earliness and tardiness cost parameters, i.e., $\alpha_j = \alpha$ and $\beta_j = \beta, j = 1, 2, \ldots, n$, and showed that for this special case the following *Shortest Variance First* (SVF) rule is optimal: *Sequence the jobs in the increasing order of the variances of their processing durations*. We will henceforth refer to the "symmetric" version of problem SET, in which all jobs have common cost parameters (namely, α and β), as **Problem SSET**. Formally, problem SSET is defined as follows:

$$\min_{\boldsymbol{\pi}, \mathbf{d}^\pi} \quad F\big(\boldsymbol{\pi}, \mathbf{d}^\pi\big) \quad = \quad \min_{\boldsymbol{\pi}, \mathbf{d}^\pi} \quad \sum_{j=1}^{n} \mathbb{E}[\alpha E_j^\pi + \beta T_j^\pi]. \tag{SSET}$$

In this paper, we study problem SSET and establish the optimality of the SVF rule for more general distributions. Specifically, we show that the SVF sequence and appropriately chosen due-dates form an optimal solution to problem SSET under the dilation order of the random processing durations. It is easy to establish that normally distributed random variables, arranged according to the SVF rule, are in dilation order. Thus, our result generalizes the result in Baker (2014) for problem SSET.

Problem SSET has an interesting application in surgical scheduling, and is closely related to the classical Appointment Scheduling Problem in that domain. In Section 2, we discuss this application and connection. Section 3 summarizes the relevant literature. Section 4 contains our analysis of problem SSET under dilation ordering of the random processing durations and our result. Section 5 establishes the validity of our main result for two extensions of problem SSET. Section 6 analyzes the impact of the variability of the processing durations of the jobs on the optimal cost. Section 7 offers some useful directions for future research.

2 Application to Surgical Scheduling

The well-known Appointment Scheduling Problem (ASP) is one of deciding the sequence in which a set of surgeries is performed and the appointment times given to the corresponding patients, with the objective of minimizing the sum of the expected waiting costs (of the patients) and the expected idling cost (of the resources, including the operating room (OR) and the team of doctors). A standard assumption in this literature is that a surgery is not allowed to start earlier than the scheduled appointment time, while idling of the resources is permitted. We will abbreviate this problem by ASwE. A variant of this problem is when the surgeries are allowed to start earlier than the scheduled appointment time, but idling of the resources is not allowed. We will abbreviate this problem by ASwI. Both problems are extreme representations of reality and the understanding of these special cases is valuable; Pinedo (2009) discusses both these problems.

We now proceed to define the problems ASwE and ASwI formally. Consider n patients, indexed by $j = 1, 2, \ldots, n$, each of whom needs to undergo a surgery in an OR. The duration of each surgery is random, with a known distribution. Let Z_j denote the random duration of the surgery for patient j, $j = 1, 2, \ldots, n$. The durations of the surgeries are assumed to be independent of each other. We denote the current time by 0 (i.e., the earliest time a surgery can start is 0).

Let $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_n)$ be the sequence in which the surgeries are performed and let $\mathbf{A}^\pi = (A_1^\pi, A_2^\pi, \ldots, A_n^\pi)$ be the corresponding appointment times. Let S_j^π denote the actual (or realized) start time of the surgery of patient π_j, $j = 1, 2, \ldots, n$. We consider the following costs:

- *Waiting cost* of a patient, denoted by c_W per unit time, incurred when the patient waits for the start of his surgery after the scheduled appointment time. The waiting cost of patient π_j is $c_W[S_j^\pi - A_j^\pi]^+$, $j = 1, 2, \ldots, n$.

- *Idling cost* of the OR, denoted by c_I per unit time, incurred due to the non-usage of the OR. The idling cost incurred between the end of the surgery of patient π_j and the start of the surgery of patient π_{j+1} is $c_I[S_{j+1}^\pi - (S_j^\pi + Z_{\pi_j})]^+$.

- *Earliness penalty* for a patient, denoted by c_E per unit time, incurred in advancing the start time of the patient's surgery. The earliness penalty incurred for patient π_j is $c_E[A_j^\pi - S_j^\pi]^+$.

The total cost corresponding to $(\boldsymbol{\pi}, \mathbf{A}^\pi)$ is the sum of the three costs mentioned above for all the patients. The goal is to obtain a sequence in which the surgeries are performed and the appointment schedule, with the objective of minimizing the total expected cost; such a schedule is referred to as an *optimal schedule*. The corresponding sequence is an *optimal sequence*. Note that problem ASwE corresponds to the following choice of the cost parameters: $c_E = \infty$, $c_W < \infty, c_I < \infty$, while the variant ASwI corresponds to $c_E < \infty, c_W < \infty, c_I = \infty$. We formally define problems ASwE and ASwI below:

$$\textbf{(ASwE)} \qquad \min_{\boldsymbol{\pi}, A^\pi \in \mathbb{R}_+^n} \quad \mathbb{E}\Bigg[\sum_{j=1}^{n} \big[c_W(S_j^\pi - A_j^\pi)^+ + c_I(A_j^\pi - S_{j-1}^\pi - Z_{\pi_{j-1}})^+\big]\Bigg], \text{ where}$$
$$S_0^\pi = Z_{\pi_0} = 0 \text{ and } S_{j+1}^\pi = \max(A_{j+1}^\pi, S_j^\pi + Z_{\pi_j}), i = 0, 1, \ldots, n-1.$$

$$\textbf{(ASwI)} \qquad \min_{\boldsymbol{\pi}, A^\pi \in \mathbb{R}_+^n} \quad \mathbb{E}\Bigg[\sum_{j=1}^{n} \big[c_W(S_j^\pi - A_j^\pi)^+ + c_E(A_j^\pi - S_j^\pi)^+\big]\Bigg], \text{ where}$$
$$S_1^\pi = 0 \text{ and } S_{j+1}^\pi = S_j^\pi + Z_{\pi_j} = \textstyle\sum_{i=1}^{j} Z_{\pi_i}, j = 1, 2, \ldots, n-1.$$

An outpatient setting – where patients arrive for surgeries from outside the hospital – is the most relevant one for these two problems. It is in this setting that the notions of waiting and earliness of the patients – and their impact on customer service metrics at hospitals – become more meaningful; see, e.g., Stempniak (2013), Globerman (2013), Klassen and Yoogalingam (2009), Pinedo (2009), and Cayirli and Veral (2003). The importance of the variant ASwI is based on the fact that surgeons and operating rooms are extremely valuable resources for hospitals; see, e.g., Steele (2014), Macario (2010) and Gupta (2007); therefore, minimizing the idle time of these resources is a priority. Indeed, the practice of starting surgeries earlier than scheduled to avoid loss of the surgeon's time is well-accepted; we provide the following illustrative quotes:

- "*Sometimes your surgeon may request you come earlier than scheduled, so please try and be available by telephone the day of your surgery.*"

  – Institute for Orthopaedic Surgery, Las Vegas, NV. (IOS, 2014)

- "*We do ask patients to arrive early for their operation in order that we can carry out any checks and also because there is inevitably some uncertainty around how long each procedure will take. It may therefore be that an operation may take place earlier than scheduled, should another be completed more quickly than expected ...*"

  – Chris Watt, Director of Performance and Delivery, Harrogate and District Hospital, Yorkshire, UK. (Watt, 2014)

Connection to problem SSET: Consider a more generalized version of problem SSET (introduced in Section 1), where the objective function is a non-negative linear combination of the expected costs of all the jobs, i.e. $F(\boldsymbol{\pi}, \mathbf{d}^\pi) = \sum_{j=1}^{n} a_j \mathbb{E}[\alpha(d_j^\pi - C_j^\pi)^+ + \beta(C_j^\pi - d_j^\pi)^+]$, where $a_j \geq 0$, for all j, $j = 1, 2, \ldots, n$. The special case of this problem corresponding to $a_j = 1$ for $j = 1, 2, \ldots, n-1$, and $a_n = 0$, is identical to problem ASwI. To see this, it is sufficient to make the following observations: Consider an arbitrary sequence $\boldsymbol{\pi}$ that corresponds to a sequence of patients in ASwI and a sequence of jobs in the special case defined above. Then,

- The appointment time A_{j+1}^π of patient π_{j+1} in ASwI corresponds to the due-date d_j^π of job π_j in problem SSET, i.e., $A_{j+1}^\pi \equiv d_j^\pi$.

- The random start time S_{j+1}^π of the surgery of patient π_{j+1} in ASwI corresponds to the random completion time C_j^π of job π_j in problem SSET, i.e., $S_{j+1}^\pi \equiv C_j^\pi$.

- The cost parameter c_E (resp., c_W) for earliness (resp., waiting) in problem ASwI corresponds to the cost parameter α (resp., β) for earliness (resp., tardiness) in problem SSET, i.e., $c_E \equiv \alpha$ and $c_W \equiv \beta$.

Throughout this paper, we will analyze problem SSET (as defined in Section 1). However, all our results remain valid for the generalized version above and, therefore, also for problem ASwI.

There exists a considerable amount of literature that investigates the optimality of the SVF rule for problem ASwE, see Section 3. In general, however, this problem has remained open. Our main result, Theorem 1, establishes the optimality of the SVF rule for the variant ASwI.

3 Related Literature

We briefly summarize the literature on problem SET. The deterministic version of this problem has received significant attention; we refer the reader to the review paper by Baker and Scudder (1990) and the textbook by Pinedo (2011). Cheng (1991) is one of the first to address the problem with random processing times. For each job, he models the total cost as the sum of two components: (a) a function of the length of the assigned due-date for that job and (b) a function of the deviation of the completion time of that job from its due-date. For this problem, he derives analytical expressions of the optimal due-dates for a given processing sequence and proposes a sorting-based algorithm to find the optimal job sequence under certain simplifying assumptions. Soroush (1999) develops two effective heuristics for problem SET and reports their computational performance. Xia et. al. (2008) propose an effective heuristic for a variant of problem SSET in which the objective includes, for each job, an additional cost penalty that is proportional to the length of its due-date. As mentioned earlier, Baker (2014) studies problem SET assuming normally distributed processing times and proposes a branch-and-bound algorithm to obtain optimal solutions. He also establishes the optimality of the SVF rule for the special case of problem SSET with normally distributed processing times.

We now summarize the investigations on problem ASwE. Weiss (1990) is arguably the first study to analyze this problem. For the special case of two patients (i.e., $n = 2$), he shows that scheduling the surgery with the lower variance first is optimal under both uniform and exponential surgery durations. This study also conjectures that the SVF rule might be optimal in general (i.e., for $n \geq 3$). Gupta (2007) investigates the optimal sequence of patients under the convex ordering of the random surgery durations. He proves the optimality of the SVF rule for $n = 2$, but reports that attempts to generalize the SVF rule to larger values of n have not been fruitful. In a recent paper, Kong et al. (2014) investigate the sequencing problem under the assumption that the allowance (difference between successive appointment times) for every surgery equals its mean duration. They produce an example in which the SVF sequence is strictly sub-optimal. Moreover, they show that the SVF rule is optimal (when each allowance equals the expected duration) under a set of assumptions, which includes the assumption that the surgery durations can be ordered with respect to the likelihood ratio order. While Kong et al. (2014) focus on the sequencing problem for a given allowance vector (equal to the mean of the surgery durations), Robinson and Chen (2003) focus on the problem of finding the optimal allowances for a given sequence of patients. Mak et al. (2013) consider a robust min-max variant of the classical problem that seeks a sequence and corresponding appointment times to minimize the maximum expected value of the total waiting and overtime costs over all distributions with given first and second moments. Under some technical conditions, it is shown that the SVF rule is optimal for this variant.

4 Main Result: Optimality of the SVF Rule for Problem SSET under Dilation Ordering

We start by defining the well-known notions of the convex order and the dilation order of random variables, see e.g., Shaked and Shantikumar (1994), and then state and prove our result.

Definition 1 (Convex Order) *A random variable X is said to be smaller than another random variable Y in the convex order, denoted by $X \leq_{cx} Y$, if $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$ for all convex functions $f : \mathbb{R} \to \mathbb{R}$, provided the expectations exist.*

Definition 2 (Dilation Order) *A random variable X with a finite mean is said to be smaller than another random variable Y with a finite mean in the dilation order, denoted by $X \leq_{dil} Y$, if $\big[X - \mathbb{E}[X]\big] \leq_{cx} \big[Y - \mathbb{E}[Y]\big]$.*

Theorem 1 (Optimal Solution) *Let p_j and μ_j denote, respectively, the random processing duration and the mean processing duration of job j, $j = 1, 2, \ldots, n$. Assume that $p_1 \leq_{dil} p_2 \leq_{dil} \ldots \leq_{dil} p_n$ (i.e., $p_1 - \mu_1 \leq_{cx} p_2 - \mu_2 \leq_{cx} \ldots \leq_{cx} p_n - \mu_n$). Let $d_j^* = \Phi_j^{-1}\left(\dfrac{\beta}{\alpha + \beta}\right)$, where Φ_j*

is the CDF of $\sum_{k=1}^{j} p_k$*. Then, the sequence* $(1, 2, \ldots, n)$ *and the due-dates* d_j^**,* $j = 1, 2, \ldots, n$*, form an optimal solution to Problem SSET.*

Proof Let $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_n)$ denote an arbitrary sequence in which jobs are processed. Let $\mathbf{d}^\pi = (d_1^\pi, d_2^\pi, \ldots, d_n^\pi)$ denote the vector of due-dates corresponding to the jobs in the sequence $\boldsymbol{\pi}$, where d_j^π denotes the due-date of job π_j. The completion time of job π_j is $C_j^\pi = \sum_{i=1}^{j} p_{\pi_i}$.

Next, for any $x \in \mathbb{R}$, let $f(x) = \alpha(-x)^+ + \beta(x)^+$. It is easy to see that $\mathbb{E}[f(C_j^\pi - d_j^\pi)]$ is the expected cost corresponding to job π_j. For any $\mathbf{d} = (d_1, d_2, \ldots, d_n)$, let

$$F(\boldsymbol{\pi}, \mathbf{d}) = \sum_{j=1}^{n} \mathbb{E}[f(C_j^\pi - d_j)]. \tag{1}$$

Thus, $F(\boldsymbol{\pi}, \mathbf{d}^\pi)$ denotes the expected cost corresponding to $(\boldsymbol{\pi}, \mathbf{d}^\pi)$.

Let $\hat{\boldsymbol{\pi}}$ be the special sequence $(1, 2, \ldots, n)$, which is the SVF sequence. We are required to show that

$$\min_{\mathbf{d}} \quad F(\hat{\boldsymbol{\pi}}, \mathbf{d}) \quad \leq \quad \min_{\mathbf{d}} \quad F(\boldsymbol{\pi}, \mathbf{d}). \tag{2}$$

We will prove (2) by showing that for any vector $\mathbf{d}^\pi$ of due-dates, there exists a carefully chosen vector $\mathbf{d}^{\hat{\pi}}$ such that $F(\hat{\boldsymbol{\pi}}, \mathbf{d}^{\hat{\pi}}) \leq F(\boldsymbol{\pi}, \mathbf{d}^\pi)$. In fact, we will establish the stronger claim that for every j, the cost corresponding to job $\hat{\pi}_j$ under $(\hat{\boldsymbol{\pi}}, \mathbf{d}^{\hat{\pi}})$ is less than the cost corresponding to job π_j under $(\boldsymbol{\pi}, \mathbf{d}^\pi)$. That is, we will show that

$$\mathbb{E}[f(C_j^{\hat{\pi}} - d_j^{\hat{\pi}})] \leq \mathbb{E}[f(C_j^\pi - d_j^\pi)] \quad \forall j = 1, 2, \ldots, n. \tag{3}$$

We proceed to define a due-date vector $\mathbf{d}^{\hat{\pi}}$ that satisfies (3).

We define the following notation. Let

$$\begin{aligned} U_j &= \sum_{k=1}^{j} (\mu_{\hat{\pi}_k} - \mu_{\pi_k}), \quad j = 1, 2, \ldots, n, \quad \text{and} \\ \mathbf{U} &= (U_1, U_2, \ldots, U_n). \end{aligned}$$

Now, define $\mathbf{d}^{\hat{\pi}}$ as follows: $\mathbf{d}^{\hat{\pi}} = \mathbf{d}^\pi + \mathbf{U}$. Corresponding to this vector of due-dates, the expected cost of job $\hat{\pi}_j$ is

$$\mathbb{E}\Big[f(C_j^{\hat{\pi}} - d_j^{\hat{\pi}})\Big] = \mathbb{E}\Big[f(C_j^{\hat{\pi}} - U_j - d_j^\pi)\Big]. \tag{4}$$

Thus, to show (3), it only remains to show that

$$\mathbb{E}\Big[f(C_j^{\hat{\pi}} - U_j - d_j^{\pi})\Big] \le \mathbb{E}\Big[f(C_j^{\pi} - d_j^{\pi})\Big]. \tag{5}$$

Since f is convex, it is sufficient to prove that $C_j^{\hat{\pi}} - U_j - d_j^{\pi} \le_{cx} C_j^{\pi} - d_j^{\pi}$. That is,

$$\sum_{k=1}^{j} p_{\hat{\pi}_k} - U_j - d_j^{\pi} \le_{cx} \sum_{k=1}^{j} p_{\pi_k} - d_j^{\pi}. \tag{6}$$

Let $\mathcal{I}_{\hat{\pi}} = \{1, 2, \ldots, j\}$, $\mathcal{I}_{\pi} = \{\pi_1, \pi_2, \ldots, \pi_j\}$ and $\mathcal{K} = \mathcal{I}_{\hat{\pi}} \cap \mathcal{I}_{\pi}$. Then,

$$\sum_{k=1}^{j} p_{\hat{\pi}_k} - U_j - d_j^{\pi} = \sum_{k \in \mathcal{I}_{\hat{\pi}} \setminus \mathcal{K}} (p_k - \mu_k) + \sum_{k \in \mathcal{K}} (p_k - \mu_k) + \sum_{k \in \mathcal{I}_{\pi}} \mu_k - d_j^{\pi} \tag{7}$$

$$\text{and } \sum_{k=1}^{j} p_{\pi_k} - d_j^{\pi} = \sum_{k \in \mathcal{I}_{\pi} \setminus \mathcal{K}} (p_k - \mu_k) + \sum_{k \in \mathcal{K}} (p_k - \mu_k) + \sum_{k \in \mathcal{I}_{\pi}} \mu_k - d_j^{\pi}. \tag{8}$$

Since $\mathcal{I}_{\hat{\pi}} = \{1, 2, \ldots, j\}$ is the set of the j "smallest jobs" in the dilation order, it is easy to see that $\sum_{k \in \mathcal{I}_{\hat{\pi}} \setminus \mathcal{K}} (p_k - \mu_k) \le_{cx} \sum_{k \in \mathcal{I}_{\pi} \setminus \mathcal{K}} (p_k - \mu_k)$. This observation, along with (7), (8), and the assumed independence of the processing durations, implies (6) as required.

Note : It is possible that the vector $\mathbf{d}^{\hat{\pi}}$ contains some negative due-dates. In that case, it is easy to show that $F(\hat{\pi}, (\mathbf{d}^{\hat{\pi}})^+) \le F(\hat{\pi}, \mathbf{d}^{\hat{\pi}})$, where for any vector $\mathbf{d} \in \mathbb{R}^n$, $\mathbf{d}^+ \in \mathbb{R}^n_+$ is the vector of component-wise positive parts.

We now consider the optimal due-dates corresponding to the optimal sequence $\hat{\boldsymbol{\pi}}$. The cost corresponding to the j^{th} job is given by $\mathbb{E}[\beta(\sum_{k=1}^{j} p_k - d_j^{\hat{\pi}})^+ + \alpha(d_j^{\hat{\pi}} - \sum_{k=1}^{j} p_k)^+]$. Notice that this expression is identical to the expected cost in a newsvendor problem in which the random demand is $\sum_{k=1}^{j} p_k$, the stocking level is $d_j^{\hat{\pi}}$, the overage cost parameter is α, and the underage cost parameter is β. Thus, the optimal due-dates are given by $d_j^{\hat{\pi}} = \Phi_j^{-1}\left(\frac{\beta}{\alpha + \beta}\right)$, where Φ_j is the CDF of $\sum_{k=1}^{j} p_k$. This has been noted in the literature, see, e.g., Baker (2014) and Soroush (1999). ■

A few observations on the result in Theorem 1 deserve to be mentioned:

- Notice that if $p_1 \le_{dil} p_2 \le_{dil} .... \le_{dil} p_n$, then $\mathrm{Var}(p_1) \le \mathrm{Var}(p_2) \le \ldots \le \mathrm{Var}(p_n)$. Thus, Theorem 1 guarantees that *an* SVF sequence is optimal for problem SSET when the processing durations of the jobs can be ordered in the dilation order. Furthermore, when the jobs have finite variances, *any* SVF sequence is optimal for problem SSET under the dilation ordering of the processing durations. This is because, for two real-valued random variables X and Y, if $X \leq_{dil} Y$ and $\text{Var}(X) = \text{Var}(Y)$, then $X - \text{E}[X] =_{st} Y - \text{E}[Y]$; see Theorem 2.2 of Denuit et. al. (2000). Therefore, $X \leq_{dil} Y$ and $Y \leq_{dil} X$. Finally, note that Theorem 1 implies the optimality of the SVF rule for problem SSET if the processing durations are in a convex order. This is because $p_1 \leq_{cx} p_2 \leq_{cx} \ldots \leq_{cx} p_n$ implies that $p_1 \leq_{dil} p_2 \leq_{dil} \ldots \leq_{dil} p_n$ when the means of all the jobs are finite.

- A nice feature of the result in Theorem 1 is that it does not depend on the values of the cost parameters or the exact expression of the cost. As long as the total cost is a convex function of the deviation from the scheduled due dates, the result provides a rigorous justification for the idea that "more predictable" jobs (or surgeries, in the case of the surgical scheduling application) should be scheduled earlier as opposed to "less predictable" jobs.

- Observe that Theorem 1 implies the result of Baker (2014) that the SVF rule is optimal when the processing times are normally distributed. Let $p_j \sim N(\mu_j, \sigma_j^2)$, $j = 1, 2, \ldots, n$ be independent random variables, and assume $\sigma_1^2 \leq \sigma_2^2 \leq \ldots \leq \sigma_n^2$. Then, it is easy to see that $p_1 - \mu_1 \leq_{cx} p_2 - \mu_2 \leq_{cx} \ldots \leq_{cx} p_n - \mu_n$, or $p_1 \leq_{dil} p_2 \leq_{dil} \ldots \leq_{dil} p_n$, and Theorem 1 applies.

Theorem 1 is also applicable to processing durations that follow several other families of distributions; e.g., uniform, gamma, Weibull, lognormal, and beta. For each of these families, if all the processing durations are from the same family and have the same mean, then the durations arranged in the increasing order of their variances are in convex order (and, therefore, dilation order); see Gupta (2007), and, Gupta and Cooper (2004). Another large class of distributions for the processing durations of the jobs, which are in dilation order, is the popular location-scale family of distributions and can be obtained as follows: Let μ_j and σ_j denote the mean and the standard deviation of the processing duration p_j of job j, $j = 1, 2, \ldots, n$, which is distributed as follows: $p_j \sim \mu_j + z_j\sigma_j$, where z_j's are i.i.d random variables with zero mean and unit standard deviation. In such cases, if $\sigma_1 \le \sigma_2 \le \ldots \le \sigma_n$, then $p_1 \le_{dil} p_2 \le_{dil} \cdots \le_{dil} p_n$.

5 Extensions

In this section, we consider two extensions of problem SSET, which we denote as **Problem SSET-E1** and **Problem SSET-E2**. The setting of the extensions is the same as that of problem SSET, but their objective functions are more general. We now define these two problems.

Problem SSET-E1: As in SSET, let $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_n)$ denote an arbitrary sequence in which the jobs are processed. Let $\mathbf{p}^\pi = (p_{\pi_1}, p_{\pi_2}, \ldots, p_{\pi_n})$. Let d_j^π denote the due-date of job π_j in the sequence $\boldsymbol{\pi}$ and let $\mathbf{d}^\pi = (d_1^\pi, d_2^\pi, \ldots, d_n^\pi)$. The realized completion time of job π_j in the sequence $\boldsymbol{\pi}$ is $C_j^\pi = \sum_{i=1}^{j} p_{\pi_i}$. Let the cost corresponding to job π_j be $\theta(C_j^\pi - d_j^\pi)$, where $\theta(\cdot)$ is a non-negative convex function such that $\theta(0) = 0$, i.e., the cost is 0 when there is no earliness or tardiness and the marginal earliness penalty and the marginal tardiness penalty are increasing functions. The objective is the same as that of problem SSET; i.e., minimize the sum of the costs corresponding to all jobs, with the sequence $\boldsymbol{\pi}$ and due-dates $\mathbf{d}^\pi$ as decisions. Problem SSET-E1 is formally defined as follows:

$$\min_{\boldsymbol{\pi}, \mathbf{d}^\pi} \quad F\big(\boldsymbol{\pi}, \mathbf{d}^\pi\big) \quad = \quad \min_{\boldsymbol{\pi}, \mathbf{d}^\pi} \quad \sum_{j=1}^{n} \mathbb{E}[\theta(C_j^\pi - d_j^\pi)]. \tag{SSET-E1}$$

Notice that SSET is a special case of SSET-E1 with $\theta(x) = \alpha(-x)^+ + \beta(x)^+$. The proof of the following remark is identical to that of Theorem 1.

Remark 1: Theorem 1 holds for problem SSET-E1. ■

Problem SSET-E2: This problem is identical to Problem SSET-E1, except that the cost corresponding to job π_j is given by $\psi(d_j^\pi) + \theta(C_j^\pi - d_j^\pi)$, where $\psi(x)$ is a non-decreasing function of x and $\theta(x)$ is a convex function of x. Here, the term $\psi(d_j^\pi)$ captures the cost associated with giving longer due-dates and the term $\theta(C_j^\pi - d_j^\pi)$ captures the cost of deviating from the due-dates (Cheng, 1991). Formally, problem SSET-E2 is defined as follows:

$$\min_{\boldsymbol{\pi}, \mathbf{d}^\pi} \quad F\big(\boldsymbol{\pi}, \mathbf{d}^\pi\big) \quad = \quad \min_{\boldsymbol{\pi}, \mathbf{d}^\pi} \quad \sum_{j=1}^{n} \Big[\psi(d_j^\pi) + \mathbb{E}[\theta(C_j^\pi - d_j^\pi)]\Big]. \tag{SSET-E2}$$

Problem SSET-E1 is trivially a special case of problem SSET-E2, with $\psi(x) = 0$ for all $x \in \mathbb{R}$. In the following theorem, we show the optimality of the SVF rule for problem SSET-E2, under the additional assumption that $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_n$.

Theorem 2 *Let p_j and μ_j denote the random processing duration and the mean processing duration of job j, $j = 1, 2, \ldots, n$. Assume that $p_1 \leq_{dil} p_2 \leq_{dil} \ldots \leq_{dil} p_n$ (i.e., $p_1 - \mu_1 \leq_{cx} p_2 - \mu_2 \leq_{cx} \ldots \leq_{cx} p_n - \mu_n$) and that $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_n$. Then, the sequence $(1, 2, \ldots, n)$ is an optimal sequence for Problem SSET-E2.*

Proof For any $\mathbf{d} = (d_1, d_2, \ldots, d_n)$, let

$$F(\boldsymbol{\pi}, \mathbf{d}) = \sum_{j=1}^{n} \Big[\psi(d_j) + \mathbb{E}[\theta(C_j^{\pi} - d_j)]\Big]. \tag{9}$$

Thus, $F(\boldsymbol{\pi}, \mathbf{d}^{\pi})$ is the cost corresponding to $(\boldsymbol{\pi}, \mathbf{d}^{\pi})$. Let $\hat{\boldsymbol{\pi}}$ be the special sequence $(1, 2, \ldots, n)$, which is the SVF sequence. We are required to show that

$$\min_{\mathbf{d}} \quad F(\hat{\boldsymbol{\pi}}, \mathbf{d}) \quad \leq \quad \min_{\mathbf{d}} \quad F(\boldsymbol{\pi}, \mathbf{d}). \tag{10}$$

As in the proof of Theorem 1, we prove (10) by showing the following stronger claim: for any vector $\mathbf{d}^{\pi}$ of due-dates, there exists a carefully chosen vector $\mathbf{d}^{\hat{\pi}}$ such that for every j, the cost corresponding to job $\hat{\pi}_j$ under $(\hat{\boldsymbol{\pi}}, \mathbf{d}^{\hat{\pi}})$ is less than the cost corresponding to job π_j under $(\boldsymbol{\pi}, \mathbf{d}^{\pi})$, i.e.,

$$\psi(d_j^{\hat{\pi}}) + \mathbb{E}[\theta(C^{\hat{\pi}} - d_j^{\hat{\pi}})] \leq \psi(d_j^{\pi}) + \mathbb{E}[\theta(C^{\pi} - d_j^{\pi})]. \tag{11}$$

Let $\mathbf{d}^{\hat{\pi}} = \mathbf{d}^{\pi} + \mathbf{U}$, where $\mathbf{U}$ is as defined in the proof of Theorem 1. Recall, from the proof of Theorem 1 that

$$\mathbb{E}\Big[\theta(C_j^{\hat{\pi}} - d_j^{\hat{\pi}})\Big] \leq \mathbb{E}\Big[\theta(C_j^{\pi} - d_j^{\pi})\Big]. \tag{12}$$

Moreover, the definition of $\hat{\boldsymbol{\pi}}$ and the assumption that $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_n$ implies that $\sum_{i=1}^{j} \mu_{\hat{\pi}_i} \leq \sum_{i=1}^{j} \mu_{\pi_i}$. Therefore, $d_j^{\hat{\pi}} \leq d_j^{\pi}$, for all $j = 1, 2, \ldots, n$. Since $\psi(\cdot)$ is a non-decreasing function, we have

$$\psi(d_j^{\hat{\pi}}) \leq \psi(d_j^{\pi}). \tag{13}$$

The claimed inequality in (11) follows immediately from (12) and (13). ∎

Remark 2 (Optimal Due-Dates): The optimal due-dates $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ can be obtained as follows: For each j, $j = 1, 2, \ldots, n$, d_j is the solution of $\min_{d_j} \big[\psi(d_j) + \mathbb{E}[\theta(C_j^{\hat{\pi}} - d_j)]\big]$, which is a univariate minimization problem. Moreover, if $\psi(\cdot)$ is convex, then this problem is a convex optimization problem. ■

6 Effect of Variability on Optimal Cost

In this section, we study the effect of variability of the processing durations of the jobs on the optimal cost of problem SSET-E2. We then illustrate our result using the popular *location-scale family* of distributions and the lognormal distribution for the processing durations of the jobs.

Consider two instances of Problem SSET-E2, $\mathbf{I}$ and $\hat{\mathbf{I}}$, which are defined as follows: Under instance $\mathbf{I}$, the processing duration p_j of the j^{th} job follows a distribution $G_j(\cdot)$, i.e., $p_j \sim G_j(\cdot)$, where the distributions $G_j(\cdot)$ are independent. Similarly, under instance $\hat{\mathbf{I}}$, the processing duration $\hat{p}_j$ of the j^{th} job follows a distribution $\hat{G}_j(\cdot)$, i.e., $\hat{p}_j \sim \hat{G}_j$, where the distributions $\hat{G}_j(\cdot)$ are independent. Let $\mathbb{E}[p_j] = \mu_j$ and $\mathbb{E}[\hat{p}_j] = \hat{\mu}_j$ for all j. Let $F^*(\mathbf{I})$ denote the optimal cost for instance $\mathbf{I}$ and $F^*(\hat{\mathbf{I}})$ denote the optimal cost for instance $\hat{\mathbf{I}}$. The following result compares the optimal cost under the two instances.

Theorem 3 *For all j, $j = 1, 2, \ldots, n$, assume that $\hat{\mu}_j \leq \mu_j$ and $\hat{p}_j \leq_{dil} p_j$ (i.e., $\hat{p}_j - \hat{\mu}_j \leq_{cx} p_j - \mu_j$). Then $F^*(\hat{\mathbf{I}}) \leq F^*(\mathbf{I})$. Moreover, if $\psi(\cdot) = 0$, then the assumption that $\hat{\mu}_j \leq \mu_j$ for all j can be dropped.*

Proof From the definition of Problem SSET-E2, we have

$$F^*(\mathbf{I}) = \min_{\boldsymbol{\pi}, \mathbf{d}^{\pi}} \quad \sum_{j=1}^{n} \Big[\psi(d_j^{\pi}) + \mathbb{E}[\theta(\sum_{i=1}^{j} p_{\pi_i} - d_j^{\pi})]\Big]. \tag{14}$$

We prove the following stronger claim: For any sequence $\boldsymbol{\pi}$ and any vector of due-dates $\mathbf{d}^{\pi}$ for instance $\mathbf{I}$, there exists a carefully chosen vector of due-dates $\hat{\mathbf{d}}^{\pi}$ for the same sequence $\boldsymbol{\pi}$ for instance $\hat{\mathbf{I}}$ such that for every j, the cost corresponding to job π_j under instance $\hat{\mathbf{I}}$ is less than the cost corresponding to the job π_j under the instance $\mathbf{I}$, i.e.,

$$\psi(\hat{d}_j^{\pi}) + \mathbb{E}\Big[\theta(\sum_{i=1}^{j} \hat{p}_{\pi_i} - \hat{d}_j^{\pi})\Big] \quad \leq \quad \psi(d_j^{\pi}) + \mathbb{E}\Big[\theta(\sum_{i=1}^{j} p_{\pi_i} - d_j^{\pi})\Big]. \tag{15}$$

Let $\hat{d}_j^\pi = d_j^\pi + \sum_{i=1}^{j}(\hat{\mu}_{\pi_i} - \mu_{\pi_i})$ for every j, $j = 1, 2, \ldots, n$. Therefore, we have

$$\sum_{i=1}^{j} \hat{p}_{\pi_i} - \hat{d}_j^\pi = \sum_{i=1}^{j}(\hat{p}_{\pi_i} - \hat{\mu}_{\pi_i}) + \sum_{i=1}^{j} \mu_{\pi_i} - d_j^\pi, \quad \text{and} \tag{16}$$

$$\sum_{i=1}^{j} p_{\pi_i} - d_j^\pi = \sum_{i=1}^{j}(p_{\pi_i} - \mu_{\pi_i}) + \sum_{i=1}^{j} \mu_{\pi_i} - d_j^\pi. \tag{17}$$

Since $\hat{p}_j \leq_{dil} p_j$ for every j, and the processing durations of the jobs are independent, it is easy to see that $\sum_{i=1}^{j}(\hat{p}_{\pi_i} - \hat{\mu}_{\pi_i}) \leq_{cx} \sum_{i=1}^{j}(p_{\pi_i} - \mu_{\pi_i})$. This observation, along with (16), (17), and the convexity of $\theta(\cdot)$ implies

$$\mathbb{E}\Big[\theta(\sum_{i=1}^{j} \hat{p}_{\pi_i} - \hat{d}_j^\pi)\Big] \quad \leq \quad \mathbb{E}\Big[\theta(\sum_{i=1}^{j} p_{\pi_i} - d_j^\pi)\Big]. \tag{18}$$

Moreover, the assumption that $\hat{\mu}_j \leq \mu_j$ for every j, implies that $\sum_{i=1}^{j} \hat{\mu}_{\pi_i} \leq \sum_{i=1}^{j} \mu_{\pi_i}$. Therefore, $\hat{d}_j^\pi \leq d_j^\pi$ for every j. Since $\psi(\cdot)$ is a non-decreasing function, we have

$$\psi(\hat{d}_j^\pi) \quad \leq \quad \psi(d_j^\pi). \tag{19}$$

The claimed inequality in (15) follows immediately from (18) and (19).

Note that if $\psi(\cdot) = 0$, we only need to show (18), and therefore, the assumption $\hat{\mu}_j \leq \mu_j$, for every j, can be dropped. ■

We now discuss the implication of Theorem 3 to the special case in which all processing durations are drawn from a location-scale family. Let μ_j and σ_j denote the mean and the standard deviation of the processing duration p_j of job j, $j = 1, 2, \ldots, n$, which is distributed as follows: $p_j \sim \mu_j + z_j\sigma_j$, where z_j's are i.i.d random variables with zero mean and unit standard deviation. Let $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)$ and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ denote the vector of means and standard deviations. Let $F^*(\boldsymbol{\mu}, \boldsymbol{\sigma})$ denote the optimal cost over all $(\boldsymbol{\pi}, \mathbf{d}^\pi)$, i.e.,

$$F^*(\boldsymbol{\mu}, \boldsymbol{\sigma}) \quad = \quad \min_{\boldsymbol{\pi}, \mathbf{d}^\pi} \quad \sum_{j=1}^{n} \Big[\psi(d_j^\pi) + \mathbb{E}[\theta(\sum_{i=1}^{j}(\mu_{\pi_i} + z_{\pi_i}\sigma_{\pi_i}) - d_j^\pi)]\Big].$$

Corollary 1: *For any vector of mean processing durations $\boldsymbol{\mu}$, the optimal cost, $F^*(\boldsymbol{\mu}, \boldsymbol{\sigma})$ is increasing in $\boldsymbol{\sigma}$.*

Proof Consider two random vectors $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ and $\hat{\mathbf{p}} = (\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n)$, where for every j, $j = 1, 2, \ldots, n$, p_j and $\hat{p}_j$ follow the following distribution: $p_j \sim \mu_j + z_j\sigma_j$ and $\hat{p}_j \sim \mu_j + z_j\hat{\sigma}_j$, where the z_j's are i.i.d random variables with zero mean and unit standard deviation. Assume $\hat{\sigma}_j \leq \sigma_j$ for every j (i.e., $\hat{\boldsymbol{\sigma}} \leq \boldsymbol{\sigma}$). We are required to show that

$$F^*(\boldsymbol{\mu}, \hat{\boldsymbol{\sigma}}) \quad \leq \quad F^*(\boldsymbol{\mu}, \boldsymbol{\sigma}).$$

Since $\hat{\sigma}_j \leq \sigma_j$, we have $\hat{\sigma}_j z_j \leq_{cx} \sigma_j z_j$. Thus, $\hat{p}_j \leq_{cx} p_j$ for every j. Since convex ordering implies dilation ordering, the desired result immediately follows from Theorem 3. ∎

We note that for surgical applications, lognormal distributions have been widely used to model the duration of surgeries; e.g., Stepaniak et. al. (2000), May et. al. (2000), Strum et. al. (2000). Therefore, we now study the implications of Theorem 3 for the special case in which all the processing durations are lognormally distributed.

Let the processing duration p_j of job j, $j = 1, 2, \ldots, n$, follow a lognormal distribution: $p_j \sim e^{y_j}$, where y_j, the associated normal random variable, has a mean μ_j and standard deviation σ_j, i.e., $p_j \sim e^{\mu_j + z_j\sigma_j}$, where the z_j's are i.i.d standard normal random variables. Denote the mean and variance of p_j by m_j and v_j, respectively. Note that $m_j = e^{\mu_j + \sigma_j^2/2}$ and $v_j = (e^{\sigma_j^2} - 1)\, m_j^2$. Let $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)$ and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ denote the vector of means and standard deviations of the associated normal random variables. Let $\mathbf{m} = (m_1, m_2, \ldots, m_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ denote the vector of the means and variances of the processing durations. Let $F^*(\boldsymbol{\mu}, \boldsymbol{\sigma})$ denote the optimal cost over all $(\boldsymbol{\pi}, \mathbf{d}^\pi)$, i.e.,

$$F^*(\boldsymbol{\mu}, \boldsymbol{\sigma}) \quad = \quad \min_{\boldsymbol{\pi}, \mathbf{d}^\pi} \quad \sum_{j=1}^{n} \Big[\psi(d_j^\pi) + \mathbb{E}[\theta(\sum_{i=1}^{j} e^{(\mu_{\pi_i} + z_{\pi_i}\sigma_{\pi_i})} - d_j^\pi)]\Big].$$

A result similar to Corollary 1 holds for lognormal distributions too, i.e., *for any vector of mean processing durations* $\mathbf{m}$, *the optimal cost* $F^*(\boldsymbol{\mu}, \boldsymbol{\sigma})$ *is increasing in* $\boldsymbol{\sigma}$. To see this, consider two random vectors $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ and $\hat{\mathbf{p}} = (\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n)$ where for every j, $j = 1, 2, \ldots, n$, p_j and $\hat{p}_j$ are lognormally distributed with the associated normal random variables, y_j and $\hat{y}_j$, respectively, distributed as follows: $y_j \sim N(\mu_j, \sigma_j^2)$ and $\hat{y}_j \sim N(\hat{\mu}_j, \hat{\sigma}_j^2)$. Assume $\hat{\sigma}_j \leq \sigma_j$ for every j (i.e., $\hat{\boldsymbol{\sigma}} \leq \boldsymbol{\sigma}$). It follows from Table 1.1 of Muller and Stoyan (2002) that if for every j, $\hat{\sigma}_j \leq \sigma_j$ and the means of $\hat{p}_j$ and p_j are equal (i.e., $e^{\hat{\mu}_j + \hat{\sigma}_j^2/2} = e^{\mu_j + \sigma_j^2/2}$, or equivalently,

$\hat{\mu}_j + \hat{\sigma}_j^2/2 = \mu_j + \sigma_j^2/2$), then $\hat{p}_j \leq_{cx} p_j$. Thus, it follows immediately from Theorem 3 that $F^*(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\sigma}}) \leq F^*(\boldsymbol{\mu}, \boldsymbol{\sigma})$. Also, note that for any vector of mean processing durations $\mathbf{m}$, an increase in $\boldsymbol{\sigma}$ implies an increase in the $\mathbf{v}$, and vice-versa. Thus, *for any vector of mean processing durations, the optimal cost is increasing in the variability of the processing durations.*

7 Conclusion

Motivated by applications in surgical scheduling, this paper studies a stochastic, single machine earliness/tardiness problem, with the sequence of processing of the jobs and their due-dates as decisions and the objective of minimizing the sum of earliness and tardiness costs over all jobs. Our main result for this problem is that the SVF rule is optimal under the assumption of dilation ordering of the processing durations. The effect of variability of the processing durations of the jobs on the optimal cost is also discussed.

In practice, three different costs are relevant in the scheduling of surgeries: earliness penalty, idling cost and waiting cost. Our analysis in this paper allows for arbitrary finite values of the the waiting cost and the earliness penalty parameters, but assumes that idling is not allowed (i.e., the idling cost parameter is infinite). Thus, a natural generalization would be to allow each of the three cost parameters to be finite. In this case, the Operating Room (OR) manager has to dynamically decide whether to advance the start time of the next surgery or to idle the OR, should a surgery complete sooner than expected. Another useful generalization is an "online" version of our problem, where patients arrive over time and, hence, the information of all the surgeries is not available at time 0. A useful direction for future research is to consider these generalizations in the presence of multiple operating rooms that function in parallel and share resources.

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