The Economics of Process Transparency

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We propose and analyze a novel framework to understand the role of non-instrumental information sharing in service operations management (OM), i.e., information shared by the firm not to affect consumers' actions, but to better manage their experience in the firm's process. The operations of the firm are organized as a *process*, consisting of a sequence of tasks, each of random duration. The firm shares real-time information with the consumer about the progress of their flow unit in the firm's process via a *process tracker*. The consumer is delay-sensitive and experiences gain-loss utility (loss-aversion and diminishing sensitivity) over time due to changes in beliefs about anticipated delay, as he awaits completion of the process. We analyze when providing such real-time progress information via process trackers help, or can possibly hurt a consumer. Our work draws upon the recent literature on belief-based/news utility in Economics. We find that in the presence of loss aversion alone, not sharing progress information is beneficial. In the presence of loss aversion and diminishing sensitivity, if low delays are likely, then sharing information is beneficial; otherwise, not sharing information is preferred. Our findings inform a service firm's post-sales transparency strategy.

Key words: News Utility, Process Analysis, Progress Disclosure, Information Design, Interface of Operations Management and Information Systems.

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I was fine with the way pizza used to work . . . where they'd say it'd show up in 45 minutes and it would take an hour.

— Domino's Pizza customer about the company's real-time pizza tracker (The Wall Street Journal 2017).

After I order, it's a black hole until I get my food at my door.

— Uber Eats' customer before the launch of the company's in-app order progress bar (Fast Company 2019).

1. Introduction

The process view of a firm – namely, the consideration of the firm as a process that transforms inputs to outputs through a collection of value-adding tasks performed by resources – is arguably the most fundamental idea in operations management. In this paper, we consider the post-sales operations process of a service firm. Typically, this process consists of multiple tasks. Consider the following examples from on-demand food delivery. At Domino's Pizza, the post-sales process begins after a customer places an order online. The various tasks in this process are: preparation, bake, quality check, and delivery (The Wall Street Journal 2017). Similarly, at Uber Eats, the post-sales process begins after a customer places an order in-app and the various tasks in this process are: order confirmation, preparation, worker dispatch, pickup and delivery (Fast Company 2019).

An important practical question for a post-sales process is: how should a service firm inform a delay-sensitive consumer about the progress of their flow-unit in the firm's process while they await completion? That is, how transparent should the design of a firm's post-sales process be?

One popular tool that service firms use to provide information about the progress of a consumer's flow-unit in the firm's process is a *process tracker*. In the above example, Domino's Pizza provides real-time information about the identity of the task being performed on a consumer's pizza as well as updates about the completion of tasks in the process at any time after they place an order but before receipt of their order. In Figure 1, we provide a snapshot of the pizza tracker that Domino's shares with its consumers. Similar process trackers are used by other firms, e.g., Uber Eats and the Internal Revenue Service (IRS); see Figure 2.

Figure 1 The Pizza Tracker at Domino's (The Wall Street Journal 2017). The post-sales process at Domino's (i.e., after an order is placed) is as follows: Preparation → Bake → Quality Check → Delivery. The tracker provides real-time information about a consumer's flow unit (pizza) at any time after a consumer places an order but before receipt of their flow-unit

Information about the progress of a consumer's flow-unit post-sales helps a consumer resolve uncertainty about their wait-time. Notice that such information about wait-time is *non-instrumental* to the consumer. That is, a firm's post-sales information disclosure strategy does not affect a consumer's actions, e.g., to engage in trade with the firm. Rather, firms provide such information to better manage their consumers' post-sales waiting experience.

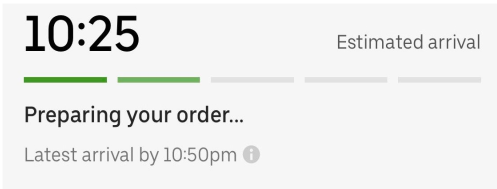


Figure 2 Process trackers in service processes. Left: Uber Eats order progress bar (Fast Company 2019). The post-sales process at Uber Eats consists of five steps: Order confirmation → Order preparation → Worker enroute to pickup → Worker pickup and confirmation → Worker delivery. Right: The Internal Revenue Service (IRS) refund status tracker (The Internal Revenue Service 2021). The post-sales process at IRS is: Return received → Refund approved → Refund sent.

Recent work in Behavioral Economics suggests that agents realize utility from non-instrumental information, called *belief-based/news* utility, in addition to their material/consumption utility. Beliefs create an *anticipation* of future consumption, and therefore directly affect an agent's utility. Consequently, agents have informational preferences, i.e., preferences over timing and structure of information (Caplin and Leahy 2001, Kőszegi and Rabin 2009, Falk and Zimmermann 2016). In the context of our work, process trackers resolve uncertainty about the consumer's wait time gradually over time, while providing no information is related to lump-sum and late resolution of uncertainty. Furthermore, providing task completion updates leads to frequent fluctuations in beliefs (relative to the lack of such progress information), and therefore hurts a loss-averse consumer (akin to Kőszegi and Rabin (2009)). In light of the above observations, a natural question arises: Does providing non-instrumental information about delay via process trackers always help, or can possibly hurt, a delay-sensitive consumer?

Beyond the theoretical motivation discussed above, our research question is of practical importance. On the one hand, firms such as Uber Eats strive to provide as much real-time information about a customer's order as possible, as evidenced by the second quote at the start of the paper and the following quote:

> *In the case of food delivery, people intuitively understand the difficulties that arise when you're trying to get hot food from a restaurant in the real world and drive it from point A to B. . . . By acknowledging some of that complexity, and being transparent about it, we can increase people's confidence a lot.*
>
> — Andy Szybalski, Global Head of Product Design, Uber Eats (Fast Company 2019)

On the other hand, providing such information can also lead to consumers forming worse/pessimistic references about the total delay in the event that some interim task takes too long. In such cases, not providing such real-time progress information might seem better, leading to fewer fluctuations in beliefs. Customers often worry when an interim task takes too long (Reddit 2020, 2019, Uber Blog 2021, Yu et al. 2022).

Our analysis provides the following insights. We highlight the contrasting roles of two forces – *loss aversion* and *diminishing sensitivity* (to news) – that play a key role in understanding the value from sharing real-time progress and task completion information. By loss aversion (to news), we mean that consumers weigh losses (bad news) more than gains (good news). By diminishing sensitivity (to news), we mean that the marginal sensitivity to news diminishes in the magnitude of the news. In other words, the consumer is less sensitive to changes in beliefs further away from the reference point. For analytical tractability, we adopt a two-part piecewise linear model for loss aversion ((2) in Section 2) and a two-part power function for diminishing sensitivity ((18) in Section 4). These functional forms allow us to capture the two forces in a parsimonious manner; we expect more elaborate models to result in qualitatively similar insights. While loss-aversion has been widely adopted in modeling consumers' reference-dependent behavior in service OM, diminishing sensitivity has been largely ignored. Sharing real-time progress information about their flow-unit leads to a greater fluctuation in the consumer's belief (relative to the absence of information sharing). Loss aversion and diminishing sensitivity (to news) affect the decision to provide real-time information in opposite directions: loss aversion favors not sharing progress information, while diminishing sensitivity favors sharing such information. The firm's strategy depends on the dominant force. In the presence of loss aversion but not diminishing sensitivity, sharing progress information hurts consumers (Theorems 1 and 2). In the presence of both loss aversion and diminishing sensitivity to news, consumers prefer multiple smaller pieces of good news over one large piece of good news, and one large piece of bad news over multiple pieces of bad news. If consumers are sufficiently insensitive to news further away from the reference, then sharing such information benefits consumers (Theorems 3 and 4).

More broadly, within the service OM literature that study the reference-dependent behavior of consumers, researchers model loss aversion but not diminishing sensitivity, perhaps due to algebraic complexity. Our work draws caution to the predictions of models that incorporate only loss aversion but not diminishing sensitivity.

1.1. Related Literature

Our paper is broadly related to multiple streams of research. The first stream of research is on delay disclosure in service OM. The extant literature exclusively focuses on settings where information about wait-time is of *instrumental* value to the consumer, i.e., information provided *pre-sales* to incentivize consumers to engage in trade with the firm, see e.g., Allon et al. (2011), Lingenbrink and Iyer (2019). In our work, we consider information about wait-time provided *post-sales* and is therefore of *non-instrumental* value to the consumer. Due to this substantial distinction, we avoid a formal discussion of this literature and refer the readers to a recent review paper on instrumental information sharing in service operations by Ibrahim (2018) and the textbook by Hassin (2016) for a comprehensive review.

The second stream of research that is closer to our work is the design and management of consumers' post-sales waiting experience. While the firm's process (sequence of tasks performed) in the settings we study is fixed, Das Gupta et al. (2016) investigate the impact of memory decay and acclimation on the design of a firm's experiential service process in settings where the firm has discretion over the sequence in which the tasks are performed. Memory decay favors positioning high service level tasks closer to the end of the process, while acclimation favors maximizing the gradient of the service level. In the context of online retail, Bray (2020) empirically analyzes the role of the timing of information disclosure (task completion updates) to non-Bayes rational consumers and demonstrates the presence of *peak-end* effect (Kahneman et al. 1993), i.e., the perceived value by consumers is higher when the task completion updates occur closer to the completion of the process. Stated differently, for a fixed total delay, customers perceive their wait in the delivery process to be better if the realizations of the task durations are such that the earlier tasks are longer and the later tasks are shorter, than if the earlier tasks are shorter and the later tasks are longer. Notably, observe that Bray (2020) fixes the firm's disclosure strategy, and compares the consumers' waiting experience across different *realizations* of the task durations. We compare the consumer's expected news utility over two disclosure strategies of the firm, where the expectation is taken over all realizations of the task durations. In particular, we do not compare consumer waiting experience across realizations of the task durations. Kumar et al. (1997) analyze the role of an explicit wait-time guarantee as a signal of the firm's reliability. They show that the explicit provision of a wait-time guarantee enhances satisfaction both during as well as at the end of wait, if the realized wait is less than the guarantee. However, if the realized wait is higher than the guarantee, the provision of a wait-time guarantee decreases satisfaction at the end of the wait. In emergency departments, where queues are partially observable, Ansari et al. (2022) empirically demonstrate that hospitals can improve patients' satisfaction by underpromising the anticipated wait and overdelivering, i.e., by quoting a wait-time estimate based on the 70th or the 90th percentile of the wait time distribution, instead of the median of the distribution.

The work that is closest to ours is Debo et al. (2022). Similar to us, they analyze a consumer who experiences gain-loss utility over time due to changes in belief about anticipated delay. In contrast, they analyze the optimal messaging strategy of a firm that is informed of the realized wait time at the start of the process. The firm has multiple opportunities to inform the consumer of the anticipated wait time while the consumer awaits completion (akin to a dynamic Bayesian persuasion model of Ely et al. (2015)). They show that if loss-aversion is dominant, the optimal strategy of the firm involves resolving greater uncertainty in the tails of the wait time distribution. If uncertainty-aversion (or risk-aversion) is dominant, the optimal strategy involves resolving greater uncertainty in the right tail of the distribution.

Our work also contributes to a third stream of research in the OM-IS interface on the design of Information Technology (IT) systems for information and process management (Kumar et al. 2018). The role of consumer informedness and its impact on a firm's information strategy is of fundamental importance. Li et al. (2014) analyze the impact of consumer informedness on a firm's information strategy. They find price informedness to be more influential for commoditized goods, but product informedness to be more influential for differentiated goods. Relatedly, Sun and Tyagi (2020) analyze the role of information provision about product fit in a distribution channel. Anand et al. (2021) empirically study sustainable process-improvement measures in high-interaction service settings (e.g., healthcare delivery) and propose a methodology for implementing such measures. In technology-intensive environments, Narayanan et al. (2020) analyze the role of task resolution and closure policies in improving technology workers' utilization, system productivity, and task outcomes.

Finally, our work also contributes to research in Economics on non-instrumental information disclosure that study an agent's preferences across two dimensions: (a) early vs. late resolution of uncertainty (sooner vs. later), and (b) one-shot vs. gradual resolution of uncertainty (clumped vs. piece-by-piece). Our work borrows the treatment of preferences towards non-instrumental information in Kőszegi and Rabin (2009), who formulate the first theory of reference-dependent utility from non-instrumental information. In their model, references are endogenous, i.e., rational beliefs at each point in time, and changes in beliefs (*news*) lead to belief-based/news utility. A central implication of their model of belief-based utility is that agents are averse to belief-fluctuations that arise due to loss aversion with respect to changes in beliefs. With regard to (a), the majority of the literature, both theoretical and experimental, finds that agents prefer earlier, rather than later, resolution of uncertainty, with a few exceptions; with regards to (b), the findings have been mixed with support for both clumped and piece-by-piece information (see Table 2 of Falk and Zimmermann (2016)).

To our knowledge, our paper is the first to explore the implications of non-instrumental information sharing and the role of news utility in better managing a consumer's post-sales waiting experience in the OM literature.

2. Model

Consider a service process comprising of n (>1) tasks in a sequence, indexed by $i \in [n]$.[1] The task durations, X_i, $i \in [n]$, are random and i.i.d.[2] Let $f(\cdot)$ denote the p.d.f (resp., p.m.f) of X_i if X_i is continuous (resp., discrete), $F(\cdot)$ denote the C.D.F., and $\overline{x}$ denote the mean. That is,

$$X_i \sim f(\cdot), F(\cdot), \quad \mathbb{E}[X_i] = \overline{x}.$$

We assume that X_i has an increasing failure rate (IFR). The assumption that task durations are IFR is common in the stochastic scheduling literature (e.g., Pinedo (2016)) and in the project management literature (e.g., Abdelkader (2004), Bendell et al. (1995)). Denote the p.d.f. (or p.m.f) (resp., C.D.F.) of $\sum_{i\in[k]} X_i$ by $f^{(k)}$ (resp., $F^{(k)}$) for any $k \in \mathbb{I}^+$. The total delay imposed by the process, denoted by D, is the sum of the task durations. That is, $D = \sum_{i\in[n]} X_i$. Therefore,

$$D \sim f^{(n)}(\cdot), F^{(n)}(\cdot), \text{ and } \mathbb{E}[D] = n\overline{x}.$$

We assume that consumers know the distributions of the durations of the tasks, i.e., the firm and the consumer share a *common prior*. Further, we also assume that the consumer uses the Bayes rule to update beliefs over time (i.e., the consumer is a rational Bayesian). Both of these are standard assumptions in games of incomplete information (Fudenberg and Tirole 1991).[3]

The consumer receives a material value of v from the completion of the service process, while the cost of waiting is normalized to 1 per unit time. Therefore, corresponding to a delay D, the consumer's material payoff (consumption utility) is:

$$U_M = v - D.$$

Therefore, a consumer rationally chooses to participate if $v > n\overline{x}$. We restrict attention to a participating consumer. Further, under the progress information sharing strategies we analyze, we show that abandonment by a participating customer before the completion of the process is irrational; see Remark D.1 in Appendix D.

[1] For any $n \in \mathbb{I}^+$, we refer to the set $\{1, 2, \ldots, n\}$ by $[n]$.

[2] In Section 5, we extend our main model to the case where task durations are independent, but not identically distributed.

[3] In practice, consumers may gather this information either based on their past experiences. Firms may also be pro-active in providing this information to customers. For example, firms may provide an interval estimate for the delay for the entire process or for each of the tasks. Domino's Pizza provides consumers with an interval estimate of the total delay. After placing an order, Domino's Pizza provides a consumer with the following message: "The entire order taking and pizza production process takes approximately 12-15 minutes" (The Wall Street Journal 2017). Similarly, the IRS states "We issue most refunds in less than 21 calendar days . . . if you filed on paper and are expecting a refund, it could take six months or more to process your return." (The Internal Revenue Service 2021).

In what follows, we first define the notion of belief-based/news utility under a progress disclosure strategy.[4] We then analyze and compare two progress disclosure strategies: At one extreme, the *opaque* strategy (denoted by OP) is an attempt to capture the traditional practice of not disclosing any progress information. At the other extreme, via the *current-task identity* strategy (denoted by CTI), the firm continuously provides real-time progress information by disclosing the identity of the task currently being performed. Our motivating examples in Section 1 – namely, the provision of real-time progress information by Domino's Pizza (Figure 1), the IRS and Uber Eats (Figure 2) – illustrate this strategy.

2.1. Progress Disclosure Strategies

Let $\boldsymbol{\pi}_t$ denote the consumer's belief (p.d.f) on D at time t and let $\boldsymbol{\Pi}$ denote the set of all distributions over D; thus, $\boldsymbol{\pi}_0 = f^{(n)}$. A progress disclosure strategy $\boldsymbol{\sigma}$ and the prior $\boldsymbol{\pi}_0$ induces a stochastic path of beliefs about the delay. We avoid a formal discussion about progress disclosure in general and restrict attention to the two progress disclosure strategies of our interest – CTI and OP – thereby simplifying our notation. Under the two progress disclosure strategies we analyze, we will show below that for a given $\boldsymbol{X} \triangleq (X_1, X_2, \ldots, X_n)$, the belief evolution is deterministic, i.e., the belief $\boldsymbol{\pi}_t^{\boldsymbol{\sigma}}$ at time t under $\boldsymbol{\sigma} \in \{\mathsf{OP}, \mathsf{CTI}\}$ for $t < D (= \sum_{i\in[n]} X_i)$ is fixed. Further, the consumer resolves all uncertainty at $t = D$ when they receive their flow unit. Thus, we require that the progress disclosure strategies resolve the uncertainty fully at $t = D$, i.e., $\boldsymbol{\pi}_D^{\boldsymbol{\sigma}} = 1 \circ D$.[5]

Define $\overline{D}_t^{\boldsymbol{\sigma}} = \mathbb{E}_{D\sim\boldsymbol{\pi}_t^{\boldsymbol{\sigma}}}[D]$. At any time $t \in (0, D)$, the consumer anticipates a material payoff of:

$$U_M\Big|_t = v - \overline{D}_t^{\boldsymbol{\sigma}}.$$

Consider a time instant t and a small interval dt. The consumer's anticipated material payoff changes by:

$$U_M\Big|_{t+dt} - U_M\Big|_t = \left(v - \overline{D}_{t+dt}^{\boldsymbol{\sigma}}\right) - \left(v - \overline{D}_t^{\boldsymbol{\sigma}}\right) = \underbrace{\overline{D}_t^{\boldsymbol{\sigma}} - \overline{D}_{t+dt}^{\boldsymbol{\sigma}}}_{\text{decrease in the } mean \text{ anticipated delay}}.$$

That is, *news* in the interval $[t, t+dt)$ corresponds to the reduction in the *mean* anticipated delay:

$$\text{News in the interval } [t, t+dt) = \overline{D}_t^{\boldsymbol{\sigma}} - \overline{D}_{t+dt}^{\boldsymbol{\sigma}}.$$

[4] We use the term *progress* disclosure strategies, instead of *delay* disclosure strategies, to avoid any confusion about the nature of information being shared. In our setting, information about delay is of non-instrumental value to the consumer.

[5] For any discrete random variable, say Y with support $\mathcal{Y}$, we denote the discrete distribution (p.m.f) with mass $p(y)$, $y \in \mathcal{Y}$ by $\sum_{y\in\mathcal{Y}} p(y) \circ y$. In particular, the degenerate distribution that places a probability mass of 1 on, say $y_1 \in \mathcal{Y}$, is denoted by $1 \circ y_1$.

We now proceed to define the belief-based utility (or news utility) under $\boldsymbol{\sigma}$. For readers unfamiliar with the notion of belief-based utility, we provide a detailed discussion in Appendix B, the canonical consumer utility model in Appendix B.1 and the main differences between our work and the extant literature in Economics in Appendix B.3. In short, belief-based/news utility corresponds to utility from *news* (non-instrumental information). Recall that the consumer is delay-sensitive. News about a decrease (resp., increase) in the anticipated delay, i.e., *good* news (resp., *bad* news) provides a positive (resp., negative) utility to the consumer at the time of provision of news in the interim while they await completion.

The belief-based utility (*news* utility), corresponding to the change in belief on D, due to the information (*news*) in the interval $[t, t+dt)$ is:

$$U_B[t,t+dt) = \mu\left(U_M\Big|_{t+dt} - U_M\Big|_{t}\right) = \mu\left(\underbrace{\overline{D}_t^{\boldsymbol{\sigma}} - \overline{D}_{t+dt}^{\boldsymbol{\sigma}}}_{\text{news in the interval } [t,t+dt)}\right), \tag{1}$$

where $\mu(\cdot)$ denotes the reference-dependent gain-loss utility model (O'Donoghue and Sprenger 2018, Duraj and He 2019).[6] That is, $\mu(\cdot)$ satisfies the following:

(A1) $\mu(\cdot)$ is continuous, increasing, differentiable (except, possibly at 0).

(A2) $\mu(0) = 0$.

(A3) Loss Aversion: $-\mu(-x) > \mu(x)$ and $\mu'(-x) > \mu'(x)$ for all $x > 0$.

(A4) Strict Diminishing Sensitivity: $\mu''(-x) > 0 > \mu''(x)$ for $x > 0$. If this inequality holds in the weak sense, we refer to this property as weak diminishing sensitivity.

Observe from (1) that the consumer's reference about delay is *endogenous* in our model. At time t, the consumer's reference is the anticipated mean delay $\overline{D}_t^{\boldsymbol{\sigma}}$. An important implication of (A3) is that the consumer is belief-fluctuation averse.

Let $\mathscr{U}^{\mathsf{B}}$ denote the set of all functions $\mu(\cdot)$ that satisfy the above assumptions. A commonly-used example for $\mu(\cdot)$ is the piecewise linear utility model (Yu et al. 2021, Ho and Zheng 2004):

$$\mu(x) = \begin{cases} \rho_P x, \text{ if } x \geq 0; \\ \rho_N x, \text{ if } x < 0. \end{cases} \tag{2}$$

[6] We adopt the changing mean-based beliefs model for the consumer's belief-based utility, consistent with recent literature on news utility (Duraj and He 2019). That is, in our model, *news* corresponds to a change in the mean anticipated delay. A more sophisticated utility model based on changing beliefs is as follows:

$$U_B[t,t+dt) = \mu\left(g(\boldsymbol{\pi}_{t+dt}^{\boldsymbol{\sigma}}) - g(\boldsymbol{\pi}_t^{\boldsymbol{\sigma}})\right)$$

where $g: \boldsymbol{\Pi} \to \Re$. In this manner, one can incorporate a consumer's preferences over other quantities, e.g., mean and variance, in their belief-based utility. See Appendix B.2 for a more detailed discussion and other forms for $U_B[t,t+dt)$.

where $0 < \rho_P < \rho_N$; thus, (2) satisfies loss aversion but not (strict) diminishing sensitivity, i.e., (A1)-(A3), but not (A4).[7] Let $\mathscr{U}^{\mathsf{PL}}$ denote the set of all piecewise linear functions as shown in (2) above.

The consumer's total utility from the process under $\boldsymbol{\sigma}$ consists of the sum of their material payoff and their belief-based utility, i.e.,

$$\begin{aligned} U^{\boldsymbol{\sigma}} &= \mathbb{E}_{D\sim\boldsymbol{\pi}_0}\left[(v-D)+\int_{t=0}^{D}\mu\left(\overline{D}_t^{\boldsymbol{\sigma}}-\overline{D}_{t+dt}^{\boldsymbol{\sigma}}\right)\right], \\ &= \underbrace{(v-n\overline{x})}_{=U_M}+\underbrace{\mathbb{E}_{D\sim\boldsymbol{\pi}_0}\left[\int_{t=0}^{D}\mu\left(\overline{D}_t^{\boldsymbol{\sigma}}-\overline{D}_{t+dt}^{\boldsymbol{\sigma}}\right)\right]}_{=U_B^{\boldsymbol{\sigma}}}. \end{aligned}$$

The first term (the material payoff U_M) is a constant and independent of $\boldsymbol{\sigma}$, i.e., it does not depend on how the news about the realized delay is shared. The second term (the belief-based utility $U_B^{\boldsymbol{\sigma}}$) depends on $\boldsymbol{\sigma}$ via the evolution of $\overline{D}_t^{\boldsymbol{\sigma}}$, the consumer's references about delay at each point of time t until completion.

2.2. Firm's Objective

The consumer does not have any actions. The firm fully internalizes the consumer's utility in the post-sales process, and maximizes $U^{\boldsymbol{\sigma}}$, i.e., the firm's problem is:

$$\max_{\boldsymbol{\sigma}\in\{\mathsf{OP},\mathsf{CTI}\}} U^{\boldsymbol{\sigma}}.$$

While we model a one-time interaction between the firm and a consumer, in reality both the firm and the consumer are long-lived, and such interactions are repeated and occur over a long horizon. The customer's post-sales experience in their current interaction affects the propensity of their future purchases. Analyzing their repeated interactions over a long horizon is analytically challenging. Therefore, we adopt a "reduced-form" approach to capture such repeated interactions by assuming that the firm is sufficiently patient, and maximizes the customer's utility in the post-sales process, although the information provided during this process is non-instrumental and does not affect the customer's decision to engage in the current trade with the firm. This simplification allows us to study the effect of information disclosure over time on the customer's post-sales waiting experience.

[7] In the rest of the paper, by diminishing sensitivity, we refer to the inequality in (A4) in the strict sense.

3. Analysis

We compare the two strategies – CTI and OP – under continuous and discrete distributions for the task durations separately; nevertheless, the intuition behind the comparison (and the dominance of one strategy over the other) remains identical. Under a piecewise linear model for the belief-based utility, we characterize the conditions under which each strategy is dominant. Surprisingly, we find that OP dominates CTI for many common distributions of task durations.[8]

3.1. Continuous Distributions

Consider the case where $f(\cdot)$ is a continuous distribution.

Opaque Strategy (OP): Under OP, the firm does not provide any update in the interim $t \in [0, D)$. Consider any time $t < D$ and a small interval dt. The consumer updates their belief based on the information that they receive in $[t, t+dt)$, that their flow unit is still in the process. Thus, the consumer's mean belief evolution is as follows:[9]

$$\overline{D}_t = \mathbb{E}_{D\sim\boldsymbol{\pi}_t^{\mathsf{OP}}}[D] = \begin{cases} t + \mathsf{MRL}_D(t), & \text{if } t < D; \\ D, & \text{if } t = D. \end{cases} \tag{3}$$

where $\mathsf{MRL}_D(t)$ denotes the mean residual life of D at t, i.e.,

$$\mathsf{MRL}_D(t) = \mathbb{E}\left[D - t | D > t\right].$$

That is, under OP, corresponding to a realization of D, the total stock of news is resolved as follows:

$$\underbrace{n\overline{x} - D}_{\text{total stock of news}} = \int_{t=0}^{D} \underbrace{-\left(1 + \mathsf{MRL}'_D(t)\right) dt}_{\textit{flow} \text{ of bad news in } t\in[0,D)} + \underbrace{\overline{D}_{D^-} - D}_{=\mathsf{MRL}_D(D),\ \textit{lump-sum} \text{ good news at } t=D}. \tag{4}$$

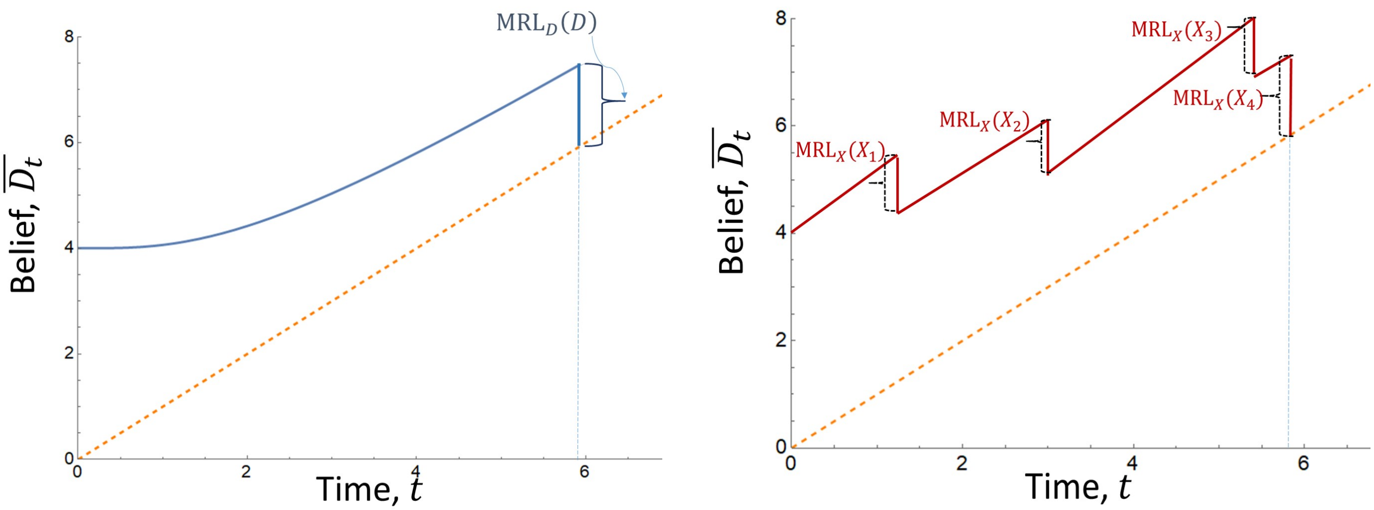
As an illustration, consider the example in the left panel of Figure 3. The consumer's mean belief, $\overline{D}_t$ increases in $t \in [0, D)$. At $t = D$, an instantaneous drop in the consumer's mean belief occurs. Stated differently, the consumer's mean belief continually worsens (a *flow of bad news*) in $t \in [0, D)$, and eventually receives a *lump-sum good news* at $t = D$. The magnitude of the good news at $t = D$, is $\mathsf{MRL}_D(D)$, the mean residual lifetime at the time of failure D. We denote the mean of this quantity by $y_{(n)}$:

$$y_{(n)} = \mathbb{E}[\mathsf{MRL}_D(D)]. \tag{5}$$

[8] For convenience, we provide a table of notation in Table A.1 in Appendix A.

[9] Precisely, the consumer's belief $\boldsymbol{\pi}_t^{\mathsf{OP}}$ (p.d.f), $t < D$ is as follows:

$$\boldsymbol{\pi}_t^{\mathsf{OP}}(x) = \begin{cases} \frac{f(x)}{\overline{F}(t)}, & \text{if } t < x < D; \\ 0, & \text{if } x \le t < D. \end{cases}, \text{ while } \boldsymbol{\pi}_D^{\mathsf{OP}} = 1 \circ D.$$



(a) OP (b) CTI

Figure 3 Belief Evolution under OP and CTI

Notes: In the above figure, we consider a process that consists of $n = 4$ tasks. The task durations are drawn from an i.i.d. exponential distribution with mean 1, i.e., $X_i \sim \mathsf{Exp}(1) \equiv f$; thus $D \sim \mathsf{Erlang}(4,1) \equiv f^{(n)} = \boldsymbol{\pi}_0$ and $\overline{D}_0 = 4$. Let the realized task durations be: $X_1 = 1.25$, $X_2 = 1.25$, $X_3 = 2.75$, and $X_4 = 0.3$. The cumulative cancelled news under OP (resp., CTI) is the mean residual life at $t = D$ (resp., sum of the mean residual life at $t = X_1, X_2, \ldots, X_4$), and is equal to $\mathsf{MRL}_D(D)$ (resp., $\sum_{i\in[4]} \mathsf{MRL}_X(X_i)$).

From Lemma F.1, it follows that $\overline{D}_t$ is increasing in t. Therefore, the expected belief-based utility under OP can be written as:

$$\begin{aligned} U_B^{\mathsf{OP}} &= \mathbb{E}_{\boldsymbol{X}}\left[\int_0^D \mu\left(-dt(1+\mathsf{MRL}_D'(t))\right) + \mu\left(\overline{D}_{D^-} - D\right)\right] \\ &= \underbrace{-\mu'(0^-)\mathbb{E}\left[\mathsf{MRL}_D(D)\right]}_{\text{Utility from the flow of bad news}} + \underbrace{\mathbb{E}\left[\mu(\mathsf{MRL}_D(D))\right]}_{\text{Utility from lump-sum good news}} . \end{aligned}$$

Suppose $\mu \in \mathscr{U}^{\mathsf{PL}}$ (i.e., μ is piecewise linear), then U_B^{OP} simplifies to

$$U_B^{\mathsf{OP}} = -\Delta_\rho y_{(n)}, \tag{6}$$

where $\Delta_\rho = \rho_N - \rho_P$. Intuitively, $y_{(n)}$ is the expected amount of *cancelled news*, i.e., the consumer's mean belief worsens, on average, by $y_{(n)}$, before improving by the same amount. Under the piecewise linear model, the consumer's belief-based utility is linear in the amount of cancelled news. Cancelled news leads to a net disutility for the consumer due to loss aversion (see (A3)). Thus, (6) follows.

Current-Task-Identity Strategy (CTI): Under CTI, the firm provides real-time updates to the consumer about the identity of the task currently being performed. In this manner, CTI resolves uncertainty about the total delay task-by-task.

For convenience, let $X_0 = 0$. Consider any time $t < D$, and a small interval dt. One of the following events occurs in the interval $[t, t+dt)$: Either the identity of the task remains the same, or an update about the completion of a task occurs. Formally, either $t \in \left(\sum_{j\in[i-1]} X_j, \sum_{j\in[i]} X_j\right)$ for some $i \in [n]$, or $t = \sum_{j\in[i]} X_j$. The consumer uses the information in this interval to update their belief on D. In particular, the information in this interval affects the consumer's belief on D only via X_i, the current task being performed. Consider the former, where $t \in \left(\sum_{j\in[i-1]} X_j, \sum_{j\in[i]} X_j\right)$ for some $i \in [n]$. The consumer's mean belief on D at t is:[10]

$$\overline{D}_t = t + \mathsf{MRL}_X\left(t - \sum_{j\in[i-1]} X_j\right) + (n-i)\overline{x}. \tag{7}$$

where $\mathsf{MRL}_X(t)$ denotes the mean residual life of X at t and $X \sim f$, i.e.,

$$\mathsf{MRL}_X(t) = \mathbb{E}\left[X - t | X > t\right].$$

Consider the latter, where $t = \sum_{j\in[i]} X_i$ for some $i \in [n]$. The consumer's mean belief on D at t is:

$$\overline{D}_t = t + (n-i)\overline{x}.$$

That is, under CTI, corresponding to a realization D, the total stock of news is resolved as follows:

$$n\overline{x} - D = \sum_{i=1}^{n}\left[\underbrace{\int_{t=\sum_{j=0}^{i-1} X_j}^{\sum_{j=0}^{i} X_j} -\left(1 + \mathsf{MRL}_X\left(t - \sum_{j=0}^{i-1} X_j\right)\right) dt}_{\text{flow of bad news in } t\in[\sum_{j=0}^{i-1} X_j, \sum_{j=0}^{i} X_j)} + \underbrace{\mathsf{MRL}_X(X_i)}_{\text{lump-sum good news at } t=X_i}\right].$$

As an illustration, consider the example in the right panel of Figure 3. The consumer's mean belief $\overline{D}_t$ is non-monotonic. Their mean belief increases continuously in $t \in [\sum_{j\in[i-1]} X_j, \sum_{j\in[i]} X_i)$ for $i \in [n]$, i.e., a *flow of bad news within each task.* At $t = \sum_{j\in[i]} X_j$, a drop in the consumer's mean belief occurs, i.e., a *lump-sum good news within each task.* The magnitude of each of these pieces of good news, on average, is $\mathbb{E}[\mathsf{MRL}_X(X)]$, the (expected) mean residual lifetime at the time of failure for X. We denote this quantity by $y_{(1)}$:

$$y_{(1)} = \mathbb{E}[\mathsf{MRL}_X(X)].$$

[10] Precisely, the consumer's belief $\boldsymbol{\pi}_t^{\mathsf{CTI}}$ (p.d.f), $t < D$ is as follows:

$$\boldsymbol{\pi}_t^{\mathsf{CTI}}(y) = \int_{\hat{x}_i=0}^{y-t} \underbrace{\frac{f(\hat{x}_i + (t - \sum_{j=0}^{i-1} X_j))}{\overline{F}(t - \sum_{j=0}^{i-1} X_j)}}_{=\mathbb{P}\left[X_i = t - (\sum_{j=0}^{i-1} X_j) + \hat{x}_i\right]} \underbrace{f^{(n-i)}(y - (t+\hat{x}_i))}_{\mathbb{P}\left[\sum_{j=i+1}^{n} X_j = y-(t+\hat{x}_i)\right]} d\hat{x}_i, \text{ where } t \in [\textstyle\sum_{j=0}^{i-1} X_j, \sum_{j=0}^{i} X_j), \text{ and } \boldsymbol{\pi}_D^{\mathsf{CTI}} = 1 \circ D.$$

Therefore, the expected belief-based utility under CTI can be written as follows:

$$U_B^{\mathsf{CTI}} = \mathbb{E}_{\boldsymbol{X}}\left[\sum_{i=1}^{n}\left[\underbrace{\int_{t=\sum_{j=0}^{i-1}X_j}^{\sum_{j=0}^{i}X_j}\mu\left(-dt\left(1+\mathsf{MRL}_X'\left(t-\sum_{j=0}^{i-1}X_j\right)\right)\right)}_{\text{Utility from flow of bad news in } t\in[\sum_{j=0}^{i-1}X_j,\sum_{j=0}^{i}X_j)} + \underbrace{\mu(\mathsf{MRL}_X(X_i))}_{\text{Utility from lump-sum good news at } t=\sum_{j=0}^{i}X_j}\right]\right]$$

$$= n\left(\underbrace{-\mu'(0^-)\mathbb{E}[\mathsf{MRL}_X(X)]+\mathbb{E}\mu(\mathsf{MRL}_X(X))}_{\text{expected belief-based utility from each task}}\right).$$

Since task durations are i.i.d, the expected belief-based utility is n times the expected belief-based utility from each task.

Suppose $\mu\in\mathscr{U}^{\mathsf{PL}}$. Then, U_B^{CTI} simplifies to

$$U_B^{\mathsf{CTI}} = -\Delta_\rho\left(ny_{(1)}\right). \tag{8}$$

Observe than $ny_{(1)}$ is the cumulative amount of *cancelled news*: within each task, the consumer's belief worsens, on average, by $y_{(1)}$, before improving by the same amount. Under the piecewise linear model, the consumer receives a net disutility that is linear in the amount of cancelled news.

Comparison: The following result compares OP and CTI under the piecewise linear model of belief-based utility. Since the consumer's belief-based utility is linear in the amount of cancelled news under the piecewise linear model, it suffices to compare the cumulative expected cancelled news under the two strategies. We adopt the following notation: $\boldsymbol{\sigma}\succ\boldsymbol{\sigma}'$ (resp., $\boldsymbol{\sigma}\prec\boldsymbol{\sigma}'$) iff $U_B^{\boldsymbol{\sigma}}>U_B^{\boldsymbol{\sigma}'}$ (resp., $U_B^{\boldsymbol{\sigma}}<U_B^{\boldsymbol{\sigma}'}$).

THEOREM 1. *Suppose $\mu(\cdot)\in\mathscr{U}^{\mathsf{PL}}$. $\mathsf{OP}\succ\mathsf{CTI}$ iff the following holds:*

$$\underbrace{ny_{(1)}}_{\text{Cancelled News under CTI}} > \underbrace{y_{(n)}}_{\text{Cancelled News under OP}}. \tag{9}$$

The inequality (9) is satisfied by the following distributions for X_i:

(a) Exponential distribution.
(b) Normal distribution.

We show the validity of (9) for these two distributions in Lemma F.6 in the Appendix. We numerically verify that (9) is also satisfied by the uniform distribution for $n\le 20$: The expressions for $y_{(n)}$ are cumbersome, and hence we provide an analytical proof for $n=2$ and use Mathematica to exactly compute the value of $y_{(n)}$ for higher values of n ($3\le n\le 20$); see Appendix G. In Appendix I, we offer numerical support for the validity of (9) for three other well-known IFR distributions – Gamma, Lognormal and Weibull – that are commonly used in the stochastic scheduling literature for modeling task durations (Pinedo 2016).

3.2. Discrete (Two-Point) Distributions

Consider the following two-point distribution for X_i with support $\{x_L, x_H\}$, $0 < x_L < x_H$:[11]

$$X_i = p \circ x_H + (1-p) \circ x_L, \text{ where } p \in (0,1). \tag{10}$$

Let $\overline{x} = \mathbb{E}[X_i] = px_H + (1-p)x_L$. Therefore, $D = \sum_{i\in[n]} X_i$ is distributed according to the following binomial distribution:

$$D \sim \sum_i q_i \circ (x_H i + x_L (n-i)) \text{ where } q_i = \binom{n}{i} p^i (1-p)^{n-i}, \text{ for } i \in \{0\} \cup [n].$$

Denote $t_i = ix_H + (n-i)x_L$ for $i \in \{0,1,\ldots,n\}$; thus, $D \in \{t_0, t_1, \ldots, t_n\}$.

Opaque Strategy: Under OP, the consumer does not receive any updates until $t = D$. Therefore, they update their belief at $t = t_i$, $i \in \{0,1,\ldots\}$, $t \le D$, at which instant they learn if $D = t_i$, or $D > t_i$; hence, they incur a belief-based utility only at these instants. Define the random variable $D_{\ge i} \triangleq D | D \ge t_i$ and its mean δ_i for $i \in \{0,1,\ldots,n\}$:

$$\begin{aligned} D_{\ge i} \triangleq D | D \ge t_i &\sim \sum_{j=i}^{n} \left(\frac{q_j}{\sum_{k=i}^{n} q_k} \circ t_j \right), \text{ and} \\ \delta_i &= \mathbb{E}\left[D_{\ge i}\right] = t_i + \Delta_x \tau_i, \end{aligned}$$

where $\Delta_x = x_H - x_L$ and $\tau_i = \frac{\sum_{j=i}^{n} q_j (j-i)}{\sum_{j=i}^{n} q_j}$.[12] It is straightforward to verify that $D_{\ge i+1} \ge_{st} D_{\ge i}$; hence, δ_i is increasing in i.

Consider a time instant $t = t_i$. The consumer learns of one of the following two events that occurs: either $D = t_i$ (occurs w.p. q_i) or $D > t_i$ (occurs w.p. $\sum_{j=i+1}^{n} q_j$). Under the former, they update their mean belief on D from δ_i to t_i (*good news*, since $\delta_i > t_i$), while under the latter, they update their mean belief from δ_i to δ_{i+1} (*bad news*, since $\delta_i < \delta_{i+1}$). The total stock of news corresponding to a realization of D, say t_{i^*}, is resolved as follows:

$$n\overline{x} - D = \underbrace{(\delta_0 - \delta_1)}_{\text{bad news at } t = t_0} + \underbrace{(\delta_1 - \delta_2)}_{\text{bad news at } t = t_1} + \ldots + \underbrace{(\delta_{i^*-1} - \delta_{i^*})}_{\text{bad news at } t = t_{i^*-1}} + \underbrace{(\delta_{i^*} - D)}_{\text{good news at } t = D (= t_{i^*})}. \tag{11}$$

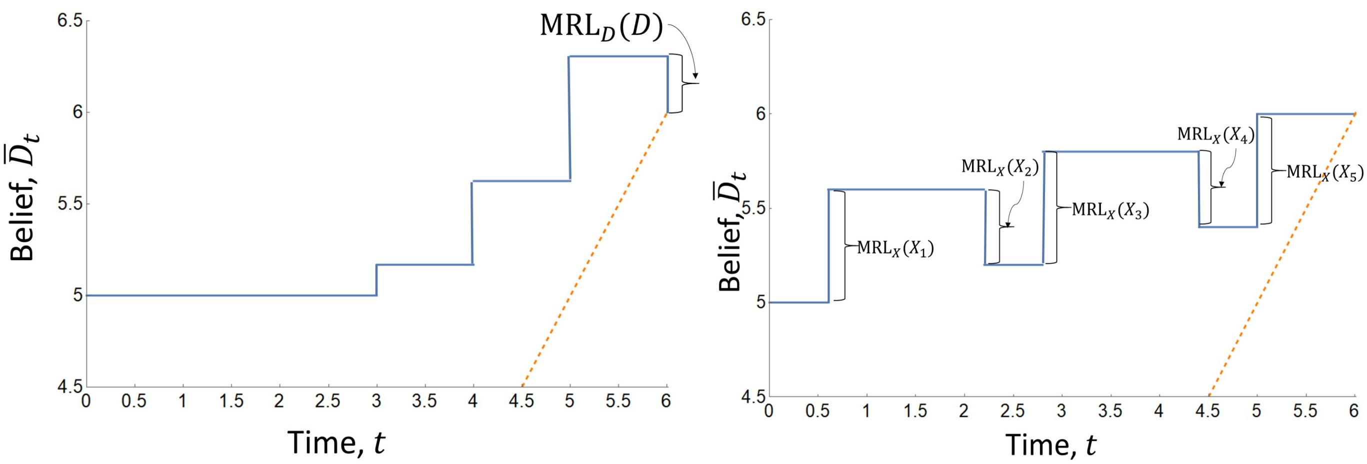
[11] For the purpose of exposition and use in subsequent analysis in Section 4, we restrict attention to a two-point distribution here. The analysis under a piecewise linear utility model and a general discrete distribution for the task durations is provided in Appendix H.

[12] Intuitively, τ_i is the *mean residual life* of a binomial random variable with parameters (n,p). Formally, let $I \sim \mathsf{Bin}(n,p)$ and $\tau_i = \mathsf{MRL}(i^-)$, i.e.,

$$\tau_i = \mathbb{E}[I - i | I \ge i].$$

Using straightforward algebra, the magnitude of good and bad news' at $t = t_i$ (depending on whether $D = t_i$, or $D > t_i$) can be further simplified as follows:

$$\text{Good news at } t = t_i \text{ (if } D = t_i\text{): } \delta_i - t_i = \tau_i \Delta_x,$$
$$\text{Bad news at } t = t_i \text{ (if } D > t_i\text{): } \delta_i - \delta_{i+1} = -\left(\frac{q_i}{\sum_{m=i+1}^{n} q_m}\right)\tau_i \Delta_x.$$



(a) OP (b) CTI

Figure 4 Belief Evolution under OP **and** CTI

Notes: In the above figure, we consider a process that consists of $n = 5$ tasks. The task durations are drawn from an i.i.d. two point distribution $X_i \sim 0.6 \circ 0.6 + 0.4 \circ 1.6$. Thus, the mean task duration is $\overline{x} = 1$ and $\Delta_x = 1$. Further, $D \sim \sum_{i=0}^{5} \binom{5}{i} 0.4^i 0.6^{5-i} \circ (1.6i + 0.6(5-i)) \equiv f^{(n)} = \boldsymbol{\pi}_0$ and $\overline{D}_0 = 5$. Let the realized task durations be: $X_1 = 1.6$, $X_2 = 0.6$, $X_3 = 1.6$, $X_4 = 0.6$, and $X_5 = 1.6$. Under OP, the belief updates occur at $t = 3, 4, 5, 6$, while under CTI, the belief updates occur at $t = 0.6, 2.2, 2.8, 4.4, 5$. The cumulative cancelled news under OP (resp., CTI) is equal to $\mathsf{MRL}_D(D)$ (resp., $\sum_{i=1}^{5} \mathsf{MRL}_X(X_i)$).

As an illustration, consider the example in the left panel of Figure 4. Under OP, the consumer's mean belief $\overline{D}_t$ increases (*worsens*) in $t \in [0, D)$, where the jumps in the mean belief occur at $t \in \{t_0, t_1, \ldots\}$. At $t = D$, the consumer experiences a decrease in their mean belief which is equal to the amount of cancelled news under OP; on average, this quantity is equal to $\mathbb{E}[\mathsf{MRL}_D(D)]$, the (expected) mean residual lifetime at the time of failure for D. We denote this quantity by $y_{(n)}$:

$$y_{(n)} = \mathbb{E}\left[\mathsf{MRL}_D(D)\right] = \sum_{i=0}^{n} q_i \underbrace{\mathsf{MRL}_D(t_i^-)}_{=\delta_i - t_i} = \sum_{i=0}^{n} q_i \tau_i \Delta_x.$$

Taken together, the consumer's expected belief-based utility from the information at $t = t_i$ is:

$$U_B^{\mathsf{OP}}[t, t+dt]\Big|_{t=t_i} = q_i \underbrace{\mu\left(\Delta_x \tau_i\right))}_{\text{Utility from } good \text{ news}} + \left(\sum_{j=i+1}^{n} q_j\right) \underbrace{\mu\left(-\Delta_x \left(\frac{q_i}{\sum_{m=i+1}^{n} q_m}\right)\tau_i\right)}_{\text{Utility from } bad \text{ news}}.$$

Therefore, the expected belief-based utility under OP, U_B^{OP}, can be written as:

$$U_B^{\mathsf{OP}} = \sum_{i=0}^{n}\left[q_i\mu\left(\Delta_x\tau_i\right) + \left(\sum_{j=i+1}^{n} q_j\right)\mu\left(-\Delta_x\left(\frac{q_i}{\sum_{m=i+1}^{n} q_m}\right)\tau_i\right)\right]. \tag{12}$$

Suppose $\mu \in \mathscr{U}^{\mathsf{PL}}$. Then, U_B^{OP} further simplifies to:

$$U_B^{\mathsf{OP}} = -\underbrace{\left(\sum_{0}^{n} q_i\tau_i\right)\Delta_x}_{=y_{(n)}}\Delta_\rho. \tag{13}$$

Current-Task Identity Strategy: Under CTI, the consumer receives an update upon each task completion, i.e., at time instants $t \in \left\{\sum_{j=0}^{i-1} X_i + x_L\right\}$, $i \in [n]$, where they learn if $X_i = x_L$ or x_H. Therefore, they incur a belief-based utility only at these time-instants.

Consider a time instant $t = \sum_{j=0}^{i-1} X_j + x_L$. The consumer learns whether $X_i = x_L$ or x_H, i.e., all uncertainty about task i's duration is resolved at t. Under the former, they update their mean belief on D from $\sum_{j=0}^{i-1} X_i + (n-i+1)\overline{x}$ to $\sum_{j=0}^{i-1} X_i + x_L + (n-i)\overline{x}$ (*good news*, since $\overline{x} > x_L$), while under the latter, they update their belief to $\sum_{j=0}^{i-1} X_i + x_H + (n-i)\overline{x}$ (*bad news*, since $\overline{x} < x_H$). Corresponding to a realization of D, the total stock of news is resolved as follows:

$$n\overline{x} - D = \underbrace{(\overline{x} - X_1)}_{\text{news resolved at } t = x_L} + \underbrace{(\overline{x} - X_2)}_{\text{news resolved at } t = X_1 + x_L} + \ldots + \underbrace{(\overline{x} - X_n)}_{\text{news resolved at } t = \sum_{j=1}^{n-1} X_j + x_L} \tag{14}$$

As an illustration, consider the right panel of Figure 4. The consumer's mean belief is non-monotonic, and depends on the uncertainty resolved in each task, i.e., the news at $t = \sum_{j=0}^{i-1} X_i + x_L$ may be good or bad, depending on the sign of $\overline{x} - X_i$. On average, the amount of cancelled news from each task is $\mathbb{E}[\mathsf{MRL}_X(X)]$, the (expected) mean residual life at the time of failure for X; we denote this quantity by $y_{(1)}$:

$$y_{(1)} = \mathbb{E}\left[\mathsf{MRL}_X(X)\right] = (1-p)\mathsf{MRL}_X(x_L^-) + p\mathsf{MRL}_X(x_H^-) = (1-p)p\Delta_x.$$

The expected belief-based utility (from the information on X_i at time $t = \sum_{j=0}^{i-1} X_i + x_L$) is:

$$U_B[t, t+dt)\Big|_{t=\sum_{j=0}^{i-1} X_i + x_L} = (1-p)\mu\left(p\Delta_x\right) + p\mu\left(-(1-p)\Delta_x\right)$$

Therefore, the total belief-based utility under CTI, U_B^{CTI}, is:

$$U_B^{\mathsf{CTI}} = n\left((1-p)\mu\left(p\Delta_x\right) + p\mu\left(-(1-p)\Delta_x\right)\right) \tag{15}$$

Suppose $\mu \in \mathscr{U}^{\mathsf{PL}}$. Then, U_B^{CTI} simplifies to:

$$U_B^{\mathsf{CTI}} = -n\underbrace{p(1-p)\Delta_x}_{y_{(1)}}\Delta_\rho. \tag{16}$$

Comparison: The following result compares OP and CTI under the piecewise linear model of belief-based utility and the two-point distribution for the task durations.

THEOREM 2. *Suppose $\mu \in \mathscr{U}^{\mathsf{PL}}$. Under the two-point distribution in (10),* $\mathsf{OP} \succ \mathsf{CTI}$ *iff:*

$$ny_{(1)} > y_{(n)}. \tag{17}$$

Observe that the conditions in (9) and (17) are identical – under discrete or continuous distributions, and the piecewise linear model, we compare the cumulative cancelled news under OP and CTI. In Appendix F, we analytically show that (17) holds for $n = 2$ and 3 (Lemma F.7). For higher values of n, the algebraic expressions are cumbersome to analyze. Nevertheless, we show that (17) holds for higher values of n ($3 \le n \le 20$) using Mathematica.[13] That is, OP dominates CTI if the task durations are drawn from an i.i.d two-point distribution and the news-utility function is piecewise linear, regardless of the parameters $(p, x_L, x_H, \rho_N, \rho_P)$.

Taken together, the substantive implications of Theorems 1 and 2 are that sharing information about the progress of a flow-unit in a service process via updates of task completion creates greater fluctuations in the consumer's beliefs. Such a strategy hurts a consumer who exhibits loss aversion to news, i.e., they are belief-fluctuation averse.

4. Diminishing Sensitivity to News

Despite the prevalence of process trackers observed in practice, our two main results thus far, Theorems 1 and 2, show that under the piecewise linear model (i.e., loss aversion (A3) but not diminishing sensitivity (A4)) and the assumptions of the respective theorems, CTI is inferior to OP. We now consider a model where a consumer exhibits loss aversion and diminishing sensitivity to news (i.e, (A3) and (A4)).[14] Due to algebraic complexity, we only study the case of a two-point distribution for the task durations.

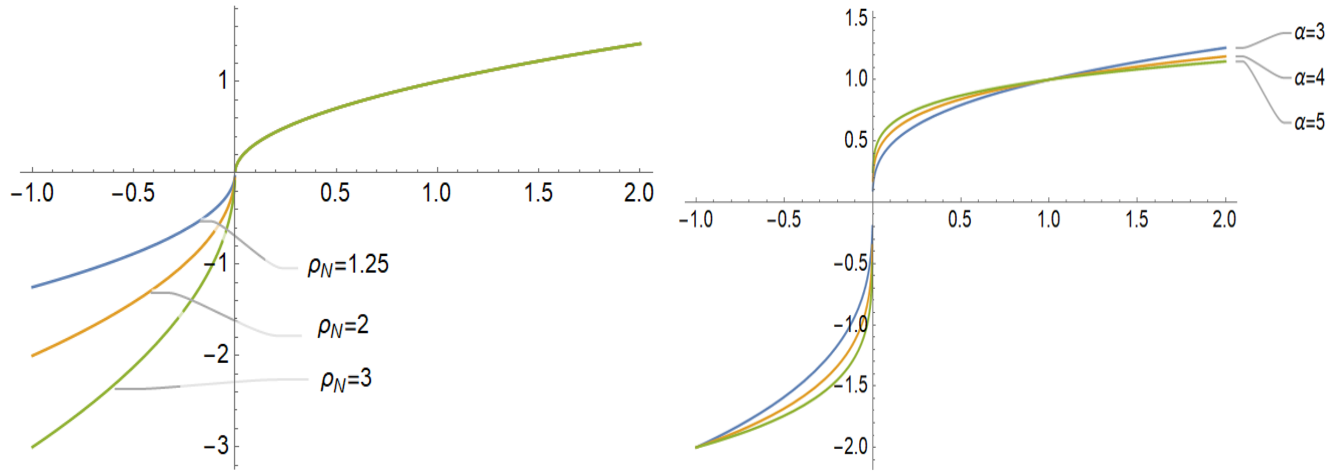
Consider the following $\mu(\cdot)$ that satisfies (A1)-(A4) (in particular, (A3) and (A4) in the strict sense):

$$\mu(x) = \begin{cases} \rho_P \sqrt[\alpha]{x}, & \text{if } x \ge 0 \\ -\rho_N \sqrt[\alpha]{-x}, & \text{if } x < 0. \end{cases} \tag{18}$$

[13] We show a weaker upper bound on $y_{(n)}$ using induction. Specifically, we show that for any $n \ge 2$ and $p \in (0,1)$, the following holds: $y_{(n)} < \frac{1}{p^n}(ny_{(1)})$ in Appendix F.1

[14] To illustrate the practical meaning of diminishing sensitivity to news about delay, consider the following example. Suppose that the mean anticipated delay changes by an amount δ. Consider a horizon, say the next $K \ge 2$ periods, where the consumer is to be informed about this change in the mean delay. Consider two scenarios:
(a) The consumer is informed of an increase in the mean delay by an amount equal to $\frac{\delta}{K}$ in each of the K periods.
(b) The consumer is informed of this increase δ in exactly one period, and no change in the other periods.
Diminishing sensitivity to news about delay refers to the idea that (a) leads to a greater psychological impact than (b). If $\delta > 0$ (an increase in the mean delay, i.e., *bad* news), then, diminishing sensitivity refers to the idea that (b) is preferred to (a). Otherwise, if $\delta < 0$, (a decrease in the mean delay, i.e., *good* news), then, (a) is preferred to (b).

where $\rho_N > \rho_P > 0$ and $\alpha > 1$. Let $\mathscr{U}^{\mathsf{DS}}$ denote the class of $\mu(\cdot)$ defined in (18). In Figure 5, we provide some examples of $\mu \in \mathscr{U}^{\mathsf{DS}}$. Without loss of generality, we assume that $\rho_P = 1 < \rho_N$. In the



(a) ρ_N (b) α

Figure 5 Role of ρ_N and α

Notes: In the left figure, we consider the following form of $\mu(\cdot)$ as shown in (18), with $\alpha = 2$. The three curves correspond to $\rho_N = 1.25, 2, 3$ respectively. In the right figure, we consider $\mu(\cdot)$ as shown in (18) with $\rho_N = 2$. The three curves correspond to $\alpha = 3, 4, 5$ respectively.

above model, ρ_N measures the degree of loss-aversion, while α measures the degree of diminishing sensitivity to news.

Substituting (18) in (12) and (15), the belief-based utility under OP and CTI is as follows:

$$U_B^{\mathsf{OP}} = \sum_{i=0}^{n} \left[q_i \sqrt[\alpha]{\Delta_x \tau_i} - \rho_N \left(\sqrt[\frac{\alpha}{\alpha-1}]{\sum_{j=i+1}^{n} q_j \sqrt[\alpha]{\Delta_x q_i \tau_i}} \right) \right] \tag{19}$$

$$U_B^{\mathsf{CTI}} = n \left((1-p) \sqrt[\alpha]{p \Delta_x} - \rho_N p \sqrt[\alpha]{(1-p) \Delta_x} \right). \tag{20}$$

While both U_B^{OP} and U_B^{CTI} above are linear in ρ_N, further analysis and comparison are difficult due to algebraic complexity. Therefore, we first analyze the special case of a two-task process below.

4.1. Two-Task Process

Consider a two-task process ($n = 2$) and $\alpha = 2$. Define the following:

$$\overline{\rho}_N = \left(2 - \sqrt{2}\right) \frac{(1-p) \left(\sqrt{2-p} + \left(\frac{\sqrt{2} - \sqrt{2-p}}{\sqrt{2}-1} \right) p \right)}{\sqrt{p} \left(2(\sqrt{2} - \sqrt{(2-p)(1-p)}) - \sqrt{2} p (3 - p - \sqrt{1-p}) \right)}. \tag{21}$$

Recall, from (19) and (20), that U_B^{OP} and U_B^{CTI} are linear in ρ_N. The following result characterizes the comparison of OP and CTI.

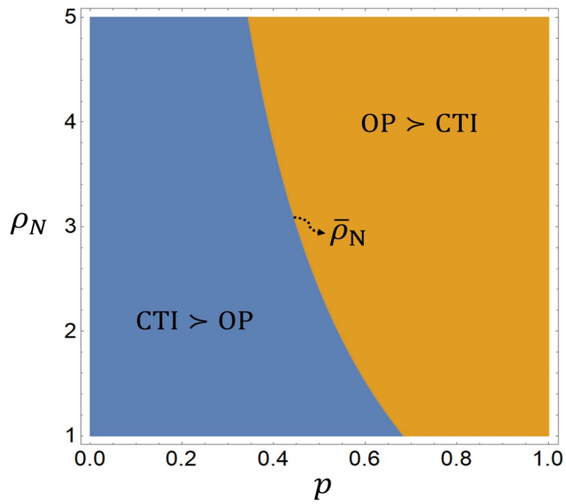


Figure 6 Comparison of OP and CTI under Loss Aversion and Diminishing Sensitivity: The x-axis denotes p, while the y-axis denotes ρ_N. Other parameters: $\alpha = n = 2$.

THEOREM 3. OP $\succ$ CTI *iff* $\rho_N > \overline{\rho}_N$.

We illustrate Theorem 3 in Figure 6. Further, we have the following structural result.

THEOREM 4. $\overline{\rho}_N$ *in* (21) *is strictly decreasing in* p. *That is, the following statements hold:*

- *If* CTI $\succ$ OP *for some* $p' \in (0,1)$*, then,* CTI $\succ$ OP *for all* $p < p'$.
- *If* OP $\succ$ CTI *for some* $p' \in (0,1)$*, then,* OP $\succ$ CTI *for all* $p > p'$.

The intuition behind Theorem 4 is as follows. Consider the following two outcomes:

- $D = 2x_L$: From (11) and (14), the total stock of news, $2\overline{x} - 2x_L$, is resolved as follows:

$$\text{Under OP: } 2\overline{x} - 2x_L = \underbrace{(2\overline{x} - 2x_L)}_{good \text{ news resolved at } t=2x_L}$$

$$\text{Under CTI: } 2\overline{x} - 2x_L = \underbrace{(\overline{x} - x_L)}_{good \text{ news resolved at } t=x_L} + \underbrace{(\overline{x} - x_L)}_{good \text{ news resolved at } t=2x_L}$$

  That is, OP resolves all uncertainty in one-shot (at $t = 2x_L$), while CTI resolves the uncertainty in two pieces – a piece $\overline{x} - x_L$ at $t = x_L$ and another piece $\overline{x} - x_L$ at $t = 2x_L$. In particular, observe that under both OP and CTI, the pieces of news are positive, i.e., *good* news (since $\overline{x} > x_L$). While both strategies provide the same stock of news, from the concavity of $\mu(\cdot)$ in $x > 0$ (good news), we have that two small pieces of good news dominates one large piece of good news, i.e., CTI $\succ$ OP.

- $D = 2x_H$: The total stock of news is, $2\overline{x} - 2x_H$, is resolved by OP and CTI as follows:

$$\text{Under OP: } 2\overline{x} - 2x_H = \underbrace{(2\overline{x} - \delta_1)}_{bad \text{ news resolved at } t=2x_L} + \underbrace{(\delta_1 - 2x_H)}_{bad \text{ news resolved at } t=x_L+x_H}$$

$$\text{Under } \mathsf{CTI}\text{: } 2\overline{x} - 2x_H = \underbrace{(\overline{x} - x_H)}_{\textit{bad} \text{ news resolved at } t = x_L} + \underbrace{(\overline{x} - x_H)}_{\textit{bad} \text{ news resolved at } t = x_L + x_H}$$

That is, OP resolves uncertainty in two pieces – a piece $2\overline{x} - \delta_1$ at $t = 2x_L$ and another piece $\delta_1 - 2x_H$ at $t = x_L + x_H$, while CTI resolves uncertainty in two equal pieces of $\overline{x} - x_H$ at $t = x_L$ and at $t = x_H + x_L$. In particular, under both OP and CTI, the pieces of news are negative, i.e., *bad* news (since $\overline{x} < x_H$). While both strategies reveal the same stock of news, the convexity of $\mu(\cdot)$ in $x < 0$ shows that two equal pieces of bad news is dominated by a large and a small piece of bad news, i.e., $\mathsf{OP} \succ \mathsf{CTI}$.

If p is small, then the more likely outcome is $D = 2x_L$; thus, CTI is the preferred choice. If p is large, the more likely outcome is $D = 2x_H$; thus, OP is the preferred choice. Stated differently, if the eventual outcome is *good* (low delay), providing multiple pieces of good news is better. If the eventual outcome is *bad* (high delay), then providing multiple pieces of bad news hurts. Therefore, the scope of the result $\mathsf{CTI} \succ \mathsf{OP}$ is decreasing in p, the likelihood of the *bad* outcome.

Role of α: Consider a more general $\mu(\cdot)$ as defined in (18).

Define $\overline{\rho}_N(\alpha)$ as follows:

$$\overline{\rho}_N(\alpha) = \frac{(1-p)\sqrt[\alpha]{p}\left(2\sqrt[\alpha]{2-p} - (1-p)\sqrt[\alpha]{2(2-p)} - 2p\right)}{p\left(2\sqrt[\alpha]{(1-p)(2-p)} - (2-p)\sqrt[\alpha]{2(1-p)^2} - p\sqrt[\alpha]{2(1-p)}\right)}. \tag{22}$$

$\overline{\rho}_N$ defined in (21) is equal to $\overline{\rho}_N(2)$ defined above. Analogous to Theorem 3, we have the following characterization:

THEOREM 5. $\mathsf{OP} \succ \mathsf{CTI}$ *iff* $\rho_N > \overline{\rho}_N(\alpha)$.

Furthermore, we observe the following:

(a) Analogous to Theorem 4, we have that $\overline{\rho}_N(\alpha)$ is decreasing in p for a given α.

(b) Over the relevant range of p (where $\overline{\rho}_N(\alpha) > 1$), we have that $\overline{\rho}_N(\alpha)$ is increasing in α for a given p.

The intuition to (a) is identical to that of Theorem 4; thus, the scope of the result $\mathsf{CTI} \succ \mathsf{OP}$ is decreasing in p. The intuition to (b) is as follows: an increase in α makes the consumer less sensitive to large pieces of news (good or bad); consequently, the scope of the result $\mathsf{CTI} \succ \mathsf{OP}$ is increasing in α. Analytical proofs of the above observations elude us due to the algebraic complexity. We illustrate these observations in Figure 7.

Theorem 4 and the observations (a) and (b) above allude to the role of diminishing sensitivity to news in the consumer's belief-based utility that leads to greater value in sharing task completion

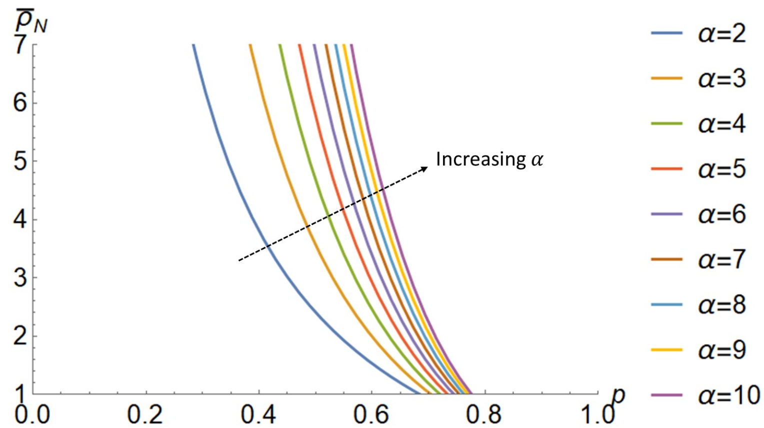


Figure 7 Effect of Higher α: $\overline{\rho}_N(\alpha)$ for $\alpha \in \{2, 3, \ldots, 10\}$. The x-axis denotes p, and the y-axis denotes $\overline{\rho}_N$. Other parameters: $n = 2$. $\overline{\rho}_N(\alpha)$ is decreasing in p for given α (Observation (a)) and increasing in α for a given p (Observation (b)).

updates in a service process. From a substantive standpoint, several papers in Behavioral Economics study the role of both loss aversion and diminishing sensitivity in the monetary domain since the seminal work by Kahneman and Tversky (1979) and in subsequent work (Kőszegi and Rabin 2009). However, almost all the work in service operations that explore reference-dependent behavior in the temporal domain consider the role of loss aversion but not diminishing sensitivity (Yu et al. 2021, Ho and Zheng 2004).

Our work highlights the contrasting roles of loss aversion and diminishing sensitivity to news in leading to opposite predictions. Loss aversion favors OP, while diminishing sensitivity favors CTI. The preferred strategy depends on the dominant economic force.

4.2. Robustness of Our Results: n-Task Processes

For robustness, we analyze processes with $n > 2$ tasks. Define the following:

$$\overline{\rho}_N(n, \alpha) = \frac{n(1-p)p^{\frac{1}{\alpha}} - \sum_{i=0}^{n} \left(q_i {\tau_i}^{\frac{1}{\alpha}}\right)}{np(1-p)^{\frac{1}{\alpha}} - \sum_{i=0}^{n} \left((q_i \tau_i)^{\frac{1}{\alpha}} \left(\sum_{j=i+1}^{n} q_j\right)^{1-\frac{1}{\alpha}}\right)}. \tag{23}$$

$\overline{\rho}_N(\alpha)$ defined in (22) is equal to $\overline{\rho}_N(2, \alpha)$ defined above, while $\overline{\rho}_N$ defined in (21) is equal to $\overline{\rho}_N(2, 2)$ defined above. From (12) and (15), we have that U_B^{OP} and U_B^{CTI} are linear in ρ_N. Analogous to Theorem 3 and 5, we have the following result from (19) and (20).

THEOREM 6. OP $\succ$ CTI *iff* $\rho_N > \overline{\rho}_N(n, \alpha)$.

An analytical investigation of $\overline{\rho}_N(n, \alpha)$ is difficult due to algebraic complexity; numerically, we observe the following:

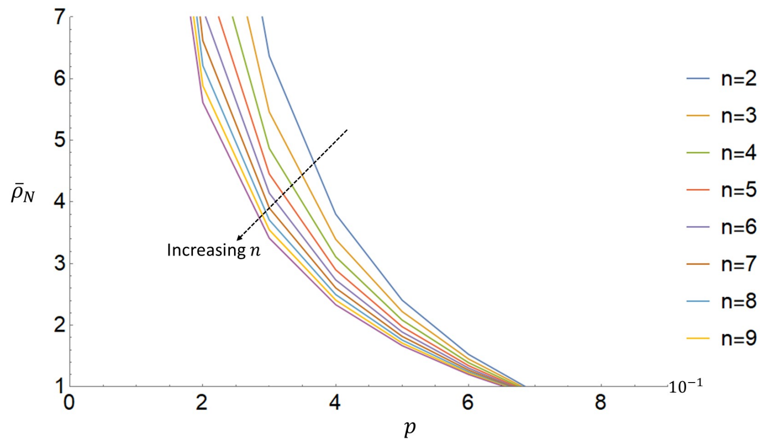


Figure 8 **Effect of Higher** n: $\overline{\rho}_N$ **for** $n \in \{2,3,\ldots,10\}$. **The** X**-axis denotes** p **while the y-axis denotes** $\overline{\rho}_N$. **Other parameters:** $\alpha = 2$.

(a) $\overline{\rho}_N(n,\alpha)$ is decreasing in p for a given n,α.

(b) $\overline{\rho}_N(n,\alpha)$ is decreasing in n for a given p,α.

Our numerical analysis, while not constituting a proof, suggests that our analytical results for the special case of $n=2$ in Theorems 3 and 4 are robust to higher values of n.

Remark 1. Consider the special case, where the customer does not exhibit loss aversion to news, but exhibits diminishing sensitivity to news (i.e., $\alpha > 1$ but $\rho_N = 1$). Consider the case where $n=2$ and $\alpha = 2$. Recall, from Theorem 4 that $\overline{\rho}_N$ is strictly decreasing in p. Define p^* as follows:

$$p^* \text{ solves } \overline{\rho}_N = 1.$$

Since $\overline{\rho}_N = \infty$ (resp., $\overline{\rho}_N = 0$) at $p=0$ (resp., $p=1$) and $\overline{\rho}_N$ is strictly decreasing, p^* is unique. By equating the RHS of (21) to 1, we obtain the value of p^*: $p^* \approxeq 0.684$. From Theorem 4, it follows that CTI dominates OP iff $p < p^*$. We can observe this in Figure 6: p^* is the point on the X-axis corresponding to $\rho_N = 1$ s.t. $\mathsf{CTI} \succ \mathsf{OP}$ for $p < p^*$.

For higher values of α and n, as mentioned above, a theoretical analysis of $\overline{\rho}_N(n,\alpha)$ is intractable. However, from our numerical analysis, we observe that $\overline{\rho}_N(n,\alpha)$ is decreasing in p. Therefore, one can obtain the threshold p^* similar to the case above, by equating the RHS in (22) (if $n=2$ and $\alpha > 1$) and (23) (if $n \geq 2, \alpha > 1$) to 1. ■

5. Extension: Non-I.I.D. Task Durations

Recall that our main model in Section 2 assumes that the task durations are i.i.d. We extend our results to the case of independent, non-identical task distributions.[15] Consider task $i \in [n]$, whose

[15] We thank the review team for suggesting this extension.

duration is denoted by X_i. Let $f_i(\cdot)$ denote the p.d.f (resp., p.m.f) of X_i if X_i is continuous (resp., discrete), $F_i(\cdot)$ denote the C.D.F., and $\overline{x}_i$ denote the mean. That is,

$$X_i \sim f_i(\cdot), F_i(\cdot), \quad \mathbb{E}[X_i] = \overline{x}_i.$$

We assume that X_i is IFR, and X_i's are independent, i.e., $X_i \perp X_j$, $i,j \in [n]$, $i \neq j$. Consider any subset $\mathcal{I} \subseteq [n]$. Denote the p.d.f. (resp., C.D.F.) of $\sum_{i\in\mathcal{I}} X_i$ by $f^{\mathcal{I}}(\cdot)$ (resp., $F^{\mathcal{I}}(\cdot)$). Therefore, the total delay $D = \sum_{i\in[n]} X_i$ is as follows:

$$D \sim f^{[n]}(\cdot), F^{[n]}(\cdot), \quad \mathbb{E}[D] = \sum_{i\in[n]} \overline{x}_i.$$

Denote the expected mean residual life at the time of failure for X_i, $i \in [n]$, and D as follows:

$$y_{(i)} = \mathbb{E}\left[\mathsf{MRL}_{X_i}(X_i)\right], \text{ and } y_{[n]} = \mathbb{E}\left[\mathsf{MRL}_D(D)\right].$$

Below, we compare the belief-based utility under the two progress disclosure strategies – OP and CTI – below.

5.1. Analysis

We first analyze the case of continuous distributions. Consider OP, where the firm does not provide any update in the interim $t \in [0, D)$. Consider any time $t < D$ and a small interval dt. The consumer's mean belief is identical to (3). Analogous to (4), under OP, corresponding to a realization of D, the total stock of news is resolved as follows:

$$\underbrace{\sum_{i\in[n]} \overline{x}_i - D}_{\text{total stock of news}} = \underbrace{-\int_{t=0}^{D^-} (1 + \mathsf{MRL}'_D(t))dt}_{\text{flow of bad news at } t<D} + \underbrace{(\overline{D}_{D^-} - D)}_{\text{lump-sum good news at } t=D}$$

Therefore, the (expected) belief-based utility under OP is:

$$\begin{aligned} U_B^{\mathsf{OP}} &= \mathbb{E}_D\left[\int_{t=0}^{D} \mu(-(1+\mathsf{MRL}'_D(t))dt) + \mu(\overline{D}_D - D)\right] \\ &= -\mu(0^-)y_{[n]} + \mathbb{E}[\mu(\mathsf{MRL}_D(D))]. \end{aligned}$$

Suppose $\mu(\cdot) \in \mathscr{U}^{\mathsf{PL}}$. Analogous to (6), we have the following:

$$U_B^{\mathsf{OP}} = -\Delta_\rho y_{[n]}. \tag{24}$$

Consider CTI, where the firm provides real-time updates to the consumer about the identity of the task currently being performed. Analogous to (7), at any time $t \in (\sum_{j\in[i-1]} X_j, \sum_{j\in[i]} X_j]$, the consumer's mean belief is:

$$\overline{D}_t = \begin{cases} t + \mathsf{MRL}_{X_i}\left(t - \sum_{j\in[i-1]} X_j\right) + \sum_{j=i+1}^{n} \overline{x}_j, & \text{if } t \in \left(\sum_{j\in[i-1]} X_j, \sum_{j\in[i]} X_j\right); \\ t + \sum_{j=i+1}^{n} \overline{x}_j, & \text{if } t = \sum_{j\in[i]} X_j \ . \end{cases}$$

where $\mathsf{MRL}_{X_i}(t) = \mathbb{E}[X_i - t | X_i > t]$. That is, under CTI, corresponding to a realization of D, the total stock of news is resolved as follows:

$$\underbrace{\sum_{i\in[n]} \overline{x}_i - D}_{\text{total stock of news}} = \sum_{i\in[n]} \left[\underbrace{\int_{t=\sum_{j\in[i-1]} X_j}^{\sum_{j\in[i]} X_j} -\left(1 + \mathsf{MRL}'_{X_i}\left(t - \sum_{j\in[i-1]} X_j\right)\right) dt}_{\text{flow of bad news in } t\in[\sum_{j\in[i-1]} X_j, \sum_{j\in[i]} X_j)} + \underbrace{\mathsf{MRL}_{X_i}(X_i)}_{\text{lump-sum good news at } t=\sum_{j\in[i]} X_j} \right].$$

Therefore, the (expected) belief-based utility under CTI is:

$$\begin{aligned} U_B^{\mathsf{CTI}} &= \mathbb{E}_{\boldsymbol{X}} \left[\int_{t=\sum_{j\in[i-1]} X_j}^{\sum_{j\in[i]} X_j} \mu \left(-\left(1 + \mathsf{MRL}'_{X_i}\left(t - \sum_{j\in[i-1]} X_j\right)\right) dt \right) + \mu(\mathsf{MRL}_{X_i}(X_i)) \right] \\ &= \sum_{i\in[n]} \left(-\mu'(0^-) y_{(i)} + \mathbb{E}[\mu(\mathsf{MRL}_{X_i}(X_i))] \right) \end{aligned}$$

Suppose $\mu(\cdot) \in \mathscr{U}^{\mathsf{PL}}$. Analogous to (8), we have the following:

$$U_B^{\mathsf{CTI}} = -\Delta_\rho \sum_{i\in[n]} y_{(i)}. \tag{25}$$

Analogous to Theorem 1, we have the following result.

THEOREM 7. *Suppose $\mu(\cdot) \in \mathscr{U}^{\mathsf{PL}}$. Then, $\mathsf{OP} \succ \mathsf{CTI}$ iff the following condition holds:*

$$\sum_{i\in[n]} y_{(i)} > y_{[n]}. \tag{26}$$

The above condition is satisfied by the exponential distribution. Further, in Appendix J, we offer numerical support for the validity of (26) for a few other IFR distributions. The intuition to the condition in the result above is identical to that in Theorem 1: $y_{[n]}$ (resp., $\sum_{i\in[n]} y_{(i)}$) is the expected amount of cancelled news under OP (resp., CTI). Since it suffices to compare the expected amount of cancelled news if $\mu(\cdot) \in \mathscr{U}^{\mathsf{PL}}$, the above result follows.

Now, consider the case of discrete (two-point) distributions. Specifically, let $\{x_{iL}, x_{iH}\}$, $0 < x_{iL} < x_{iH}$ denote the support of X_i with the following distribution:

$$X_i \sim p_i \circ x_{iH} + (1 - p_i) \circ x_{iL}. \tag{27}$$

Let $\mathcal{D}$ denote the support of D; $|\mathcal{D}| \le 2^n$. Let $\mathcal{D} = \{d_1, d_2, \dots\}$ with $d_1 < d_2 < \dots$. For notational convenience, let $\Delta_{x_i} = x_{iH} - x_{iL}$, and q_j and δ_j denote the following:

$$\begin{aligned} q_j &= f^{[n]}(d_j) = \mathbb{P}[D = d_j] \implies D \sim \sum_j q_j \circ d_j, \\ \delta_j &= \mathbb{E}[D | D \ge d_j] = \frac{\sum_{i\ge j} q_j d_j}{\sum_{i\ge j} q_i} = d_j + \mathsf{MRL}_D(d_j). \end{aligned}$$

Indeed, $\delta_1 = \sum_{i\in[n]} \overline{x}_i$. Consider OP, and a realization of D, say d_k. The total stock of news is resolved as follows:

$$\underbrace{\sum_{i\in[n]} \overline{x}_i - d_k}_{\text{total stock of news}} = \underbrace{\underbrace{(\delta_1-\delta_2)}_{\text{news resolved at } t=d_1} + \underbrace{(\delta_2-\delta_3)}_{\text{news resolved at } t=d_2} + \ldots + \underbrace{(\delta_{k-1}-\delta_k)}_{\text{news resolved at } t=d_{k-1}}}_{\text{stream of bad news}} + \underbrace{\underbrace{(\delta_k-d_k)}_{\text{news resolved at } t=d_k}}_{\text{good news}}$$

Therefore, the (expected) belief-based utility is as follows:

$$U_B^{\mathsf{OP}} = \mathbb{E}\left[\underbrace{\sum_{j\in[k-1]} \mu(\delta_j - \delta_{j+1}) + \mu(\delta_k - d_k)}_{U_B^{\mathsf{OP}}(d_k)}\right]. \tag{28}$$

Suppose $\mu(\cdot) \in \mathscr{U}^{\mathsf{PL}}$. Then, the RHS simplifies to:

$$U_B^{\mathsf{OP}} = -\Delta_\rho y_{[n]}.$$

Consider CTI. The total stock of news, corresponding to a realization of D, say d_k, is resolved as follows:

$$\underbrace{\sum_{i\in[n]} \overline{x}_i - d_k}_{\text{total stock of news}} = \underbrace{(\overline{x}_1 - X_1)}_{\text{resolved at } t=x_{1L}} + \underbrace{(\overline{x}_2 - X_2)}_{\text{resolved at } t=X_1+x_{2L}} + \ldots + \underbrace{(\overline{x}_n - X_n)}_{\text{resolved at } t=\sum_{i\in[n-1]} X_i + x_{nL}}.$$

Therefore, the (expected) belief-based utility is as follows:

$$U_B^{\mathsf{CTI}} = \mathbb{E}\left[\sum_{i\in[n]} \mu(\overline{x}_i - X_i)\right] = \sum_{i\in[n]} \left[(1-p_i)\mu(p_i\Delta_{x_i}) + p_i\mu(-(1-p_i)\Delta_{x_i})\right]. \tag{29}$$

Suppose $\mu(\cdot) \in \mathscr{U}^{\mathsf{PL}}$. Then, the RHS simplifies to:

$$U_B^{\mathsf{CTI}} = -\Delta_\rho \left(\sum_{i\in[n]} \underbrace{p_i(1-p_i)\Delta_{x_i}}_{y_{(i)}}\right) = -\Delta_\rho \sum_{i\in[n]} y_{(i)}.$$

Analogous to Theorem 2, we have the following result.

THEOREM 8. *Suppose $\mu(\cdot) \in \mathscr{U}^{\mathsf{PL}}$ and X_i is distributed as in (27). Then, $\mathsf{OP} \succ \mathsf{CTI}$ iff the following condition holds:*

$$\sum_{i\in[n]} y_{(i)} > y_{[n]}.$$

Further, Remark D.1 – abandonment by a participating consumer is irrational at $t > 0$ under OP and CTI – also holds under the non-i.i.d. setting above. The IFR property is closed under sum (see Chapter 2.5.1 of Lai and Xie (2006)). Since X_i, $i \in [n]$ are IFR, D is also IFR. Since IFR implies DMRL, D and X_i, $i \in [n]$ are DMRL. Consequently, Remark D.1 also holds under independent, non-identical, task durations, if X_i's are IFR.

5.2. Diminishing Sensitivity

Consider $\mu(\cdot)$ as shown in (18). We restrict attention to the case of $n=2$. Consider the discrete (two-point) distribution for the task durations, as shown in (27). Evaluating U_B^{OP} and U_B^{CTI} (i.e., substituting (27) in (28) and (29)), we can verify that U_B^{OP} and U_B^{CTI} depend on x_{iL}, x_{iH} only via Δ_{x_i}. To keep the analyses consistent, we assume $\Delta_{x_i} = \Delta_x$. Without l.o.g., $\Delta_x = 1$ and $\rho_P = 1$. Therefore, the support of D is:

$$\mathcal{D} = \{x_{1L} + x_{2L}, \underbrace{x_{1H} + x_{2L}}_{=x_{1L}+x_{2H}}, x_{1H} + x_{2H}\}; |\mathcal{D}| = 3.$$

Using (28) and (29), we have that both U_B^{OP} and U_B^{CTI} are linear in ρ_N. Analogous to (21), define the following:

$$\overline{\rho}_N = \frac{(1-p_1)\sqrt{p_1} + (1-p_2)\sqrt{p_2} - \sqrt{p_1+p_2}(1-p_1)(1-p_2) + \sqrt{\frac{p_1p_2}{p_1+p_2-p_1p_2}}(2p_1p_2 - p_1 - p_2)}{p_1\sqrt{1-p_1} + p_2\sqrt{1-p_2} + \sqrt{\frac{(1-p_1)(1-p_2)(p_1+p_2)}{p_1+p_2-p_1p_2}}(2p_1p_2 - p_1 - p_2) - p_1p_2\left(\sqrt{\frac{(1-p_1)(1-p_2)(p_1+p_2)}{p_1+p_2-p_1p_2}} + \sqrt{\frac{p_1p_2}{p_1+p_2-p_1p_2}}\right)}. \tag{30}$$

Observe that the RHS is symmetric in (p_1, p_2), and is identical to (21) if $p_1 = p_2$. Analogous to Theorem 3, we have the following result.

THEOREM 9. $\mathsf{OP} \succ \mathsf{CTI}$ *iff* $\rho_N > \overline{\rho}_N$.

Role of α: Under a more general $\mu(\cdot)$ as defined in (18), define the following:

$$\overline{\rho}_N(\alpha) = \frac{(1-p_1)\sqrt[\alpha]{p_1} + (1-p_2)\sqrt[\alpha]{p_2} - \sqrt[\alpha]{p_1+p_2}(1-p_1)(1-p_2) + \sqrt[\alpha]{\frac{p_1p_2}{p_1+p_2-p_1p_2}}(2p_1p_2 - p_1 - p_2)}{p_1\sqrt[\alpha]{1-p_1} + p_2\sqrt[\alpha]{1-p_2} + \sqrt[\alpha]{\frac{(1-p_1)(1-p_2)(p_1+p_2)}{p_1+p_2-p_1p_2}}(2p_1p_2 - p_1 - p_2) - p_1p_2\left(\sqrt[\alpha]{\frac{(1-p_1)(1-p_2)(p_1+p_2)}{p_1+p_2-p_1p_2}} + \sqrt[\alpha]{\frac{p_1p_2}{p_1+p_2-p_1p_2}}\right)}. \tag{31}$$

Analogous to Theorem 5, we have the following result.

THEOREM 10. $\mathsf{OP} \succ \mathsf{CTI}$ *iff* $\rho_N > \overline{\rho}_N(\alpha)$.

***n*-Task Processes:** Under higher values of n, and any Δ_{x_i}, $i \in [n]$, define:

$$\overline{\rho}_N(n, \alpha) = \frac{\sum_{i\in[n]}(1-p_i)\sqrt[\alpha]{p_i\Delta_{x_i}} - \sum_{i\in|\mathcal{D}|} q_i \sqrt[\alpha]{\delta_i - d_i}}{\sum_{i\in[n]} p_i \sqrt[\alpha]{(1-p_i)\Delta_{x_i}} - \sum_{i\in|\mathcal{D}|}(1 - \sum_{j\in[i]} q_j)\sqrt[\alpha]{\delta_{i+1} - \delta_i}}. \tag{32}$$

(32) is identical to (31) if $n = 2$ and $\Delta_{x_i} = 1$. We have the following result.

THEOREM 11. $\mathsf{OP} \succ \mathsf{CTI}$ *iff* $\rho_N > \overline{\rho}_N(n, \alpha)$.

We summarize these changes in the table below.

Table 1 Correspondence Between the Model Assumptions and Results for the I.I.D. and Non-I.I.D. Task Durations. The notation used here is summarized in Table A.1 in Appendix A.

	I.I.D. Task Distributions	Non-I.I.D. Task Distributions
Task Distributions	$X_i \sim f(\cdot), F(\cdot)$, $\mathbb{E}[X_i] = \overline{x}$	$X_i \sim f_i(\cdot), F_i(\cdot)$, $\mathbb{E}[X_i] = \overline{x}_i$
Delay Distribution	$D \sim f^{(n)}(\cdot), F^{(n)}(\cdot)$, $\mathbb{E}[D] = n\overline{x}$	$D \sim f^{[n]}(\cdot), F^{[n]}(\cdot)$, $\mathbb{E}[D] = \sum_{i\in[n]} \overline{x}_i$
Comparison of OP and CTI under Loss Aversion	Theorems 1 and 2 show OP $\succ$ CTI $\Leftrightarrow y_{(n)} < ny_{(1)}$	Theorems 7 and 8 show OP $\succ$ CTI $\Leftrightarrow$ $y_{[n]} < \sum_i y_{(i)}$
Comparison of OP and CTI under Loss Aversion and Diminishing Sensitivity	Theorems 3, 5 and 6 show OP $\succ$ CTI $\Leftrightarrow \rho_N > \overline{\rho}_N(n,\alpha)$, where $\overline{\rho}_N(n,\alpha)$ is shown in (23)	Theorems 9, 10 and 11 show OP $\succ$ CTI $\Leftrightarrow \rho_N > \overline{\rho}_N(n,\alpha)$, where $\overline{\rho}_N(n,\alpha)$ is shown in (32)

6. Substantive and Managerial Implications

From a substantive standpoint, our work demonstrates the roles of loss aversion and diminishing sensitivity to news in understanding consumers' reference-dependent utility from non-instrumental progress information. While several papers in OM study the role of loss aversion in understanding consumer behavior, diminishing sensitivity has received far less attention, perhaps due to algebraic complexity. Our work highlights the contrasting roles of loss aversion and diminishing sensitivity (to news), and how predictions of a model with loss aversion but not diminishing sensitivity can be different from those that consider both.

From a managerial standpoint, our work highlights the role of non-instrumental information about delay in better managing a consumer's post-sales waiting experience. While the extant literature in service OM focuses exclusively on whether and how a firm shares instrumental information about delay pre-sales to the consumer, our work complements this literature in proposing the first model on non-instrumental delay information post-sales. Recall, from our results in Sections 4 and 5.2, that in the presence of both loss-aversion and diminishing sensitivity, if good news is more likely, the CTI strategy is preferred; otherwise, if bad news is more likely, the OP strategy is preferred. Intuitively, the likelihood of good/bad news depends on the skewness in the distributions of task durations.

- If the distributions of the task durations are *right-skewed* (e.g., the two-point distribution[16] with a lower value of p, where p denotes the probability mass of the high task duration), then good news is more likely. Therefore, for such processes, our results indicate that disclosing real-time information via the CTI strategy is preferred.

[16] Our analysis in Sections 4 and 5.2 assumes a two-point distribution. Hence, p (the probability mass on the high duration) affects the first and the third moment. Suppose the task duration is $X \sim p \circ x_H + (1-p) \circ x_L$. Then, the

- If the distributions of the task durations are *left-skewed* (e.g., the two-point distribution with a higher value of p), then bad news is more likely. Our results indicate that OP is preferred.

From a manager's perspective, it is easy to assess whether the task durations in a process are left or right-skewed. Processes that are less prone to delay shocks are likely to follow right-skewed distributions for the task durations. Our results suggest that in such processes, the CTI strategy is preferable. Processes that are more prone to delay shocks (e.g., via disruptions to resources) are likely to follow left-skewed distributions for the task durations. Our results predict that in such processes, OP is preferable. The extent of loss-aversion and diminishing sensitivity in the consumer population and the amount of skewness in the distributions of the task durations determines which of the two strategies are dominant. For a given process, a manager can determine the extent of loss aversion and diminishing sensitivity in their consumer population based on consumers' responses to the two progress disclosure strategies.

Another insight that emerges from our work is that even if the manager has discretion over the sequence in which tasks (in a process) are performed, the sequence of tasks does not affect the comparison of OP and CTI. This is because of the following: the news utility under OP depends on the convolution of the distribution of the task durations, and the news utility under CTI is additively separable in the news utility under each task. Neither of these depend on the sequence in which tasks are performed. A manager who has discretion over the sequence in which tasks are performed can experiment over different sequences to confirm the strength of our results.

Some limitations of our work that future research can address are as follows. While our work considers two progress disclosure strategies commonly observed in practice, namely OP and CTI, future research can consider other progress disclosure strategies, depending on the information available to the firm. Suppose the firm has commitment power, i.e., the firm is able to credibly commit to an information disclosure strategy. Then, one can analyse the firm's optimal signaling strategy to inform the consumer about the delay during their wait. If the firm lacks such commitment power, then one can analyze the resulting dynamic cheap-talk game between the firm and the consumer. The consumer's belief-based utility in our model arises due to good and bad news about delay (i.e., information that increases or decreases his mean belief on delay). Future research can consider more general forms of belief-based utility, e.g., anxiety costs (Iyer and Zhong 2021), suspense and surprise (Ely et al. 2015), etc. We assume that the consumer's value from the service

skewnewss (third-centered moment) is:

$$\gamma(p) = \mathbb{E}\left[\left(\frac{X-\mu}{\sigma}\right)^3\right] = -\frac{2}{\sqrt{p(1-p)}}\left(p - \frac{1}{2}\right).$$

$\gamma(p)$ is decreasing in p; $p > p^*$ (where p^* is shown in Remark 1) is equivalent to $\gamma(p) < \gamma(p^*)$.

process is independent of the delay. Future research can consider the case of customer-intensive processes where the value from the service may be correlated with delay (Anand et al. 2011).

The two behavioral forces we consider – namely, loss-aversion and diminishing sensitivity – are two of the most frequently-studied behavioral forces, starting from the pioneering work of Kahneman and Tversky (1979). These two forces are well-understood in the literature that studies consumers' preferences to news about monetary payoffs (Kőszegi and Rabin 2009, O'Donoghue and Sprenger 2018, Duraj and He 2019). We hope that future work in Behavioral OM can help bridge this gap by measuring the strength of these behavioral forces in understanding consumers' psychology towards news about delay during their wait in the process. Beyond these two forces, future research can address other behavioral forces that can affect the decision to share real-time progress information. For example, sunk cost fallacy (or sunk cost effect) refers to the phenomenon where individuals continue an endeavor as a result of previously invested resources, e.g., time, money or effort (Thaler 1980). Another example is hyperbolic discounting, which refers to the idea that individuals' valuations of immediate payoffs fall rapidly early on, but less rapidly in the future (Rubinstein 2003). Future research can also address the case where consumers may have mis-specified beliefs, and how such mis-specification may affect the firm's decision to share real-time progress information.

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Online Appendices (Electronic Companion)

Appendix A: Table of Notation

Table A.1 Table of Notation

Symbol	Interpretation
n	Number of tasks in the process
i	Index for the tasks, $i \in \{1, 2, \ldots, n\}$
$[j]$	Set $\{1, 2, \ldots, j\}$ for any positive integer j
X_i	(Random) Duration of task i, $i \in [n]$
D	(Random) Total delay in the process, $D = \sum_{i \in [n]} X_i$
$f(\cdot)$, $F(\cdot)$ (resp., $f_i(\cdot)$, $F_i(\cdot)$)	p.d.f. and C.D.F. of X under i.i.d task durations (resp., X_i under independent, non-i.i.d. task durations)
$f^{(n)}(\cdot)$, $F^{(n)}(\cdot)$ (resp., $f^{[n]}(\cdot)$, $F^{[n]}(\cdot)$)	p.d.f. and C.D.F. of D, the n-fold convolution of X, under i.i.d task durations (resp., the sum $\sum_{i \in [n]} X_i$ under independent, non-i.i.d task durations)
$\mathsf{MRL}_Y(t)$	Mean Residual Life of a random variable Y at time t
$y_{(1)}$ (resp., $y_{(i)}$)	Expected Mean Residual Life at the Time of Failure for X under i.i.d. task durations (resp., X_i under independent, non-i.i.d. task durations)
$y_{(n)}$, (resp., $y_{[n]}$)	Expected Mean Residual Life at the Time of Failure for D under i.i.d. task durations (resp., under independent, non-i.i.d. task durations)
v	Consumer's value from the service process
OP	Opaque strategy
CTI	Current-Task Identity strategy
$\boldsymbol{\sigma}$	Progress information disclosure strategy, $\boldsymbol{\sigma} \in \{\mathsf{OP}, \mathsf{CTI}\}$
U_M	Consumer's material payoff from the service process, $U_M = v - D$
U_B	Consumer's belief-based utility (news utility)
$\mu(\cdot)$	Reference-dependent gain-loss utility function (from news)
ρ_P, ρ_N	Parameters that capture the consumer's utility from gains and losses (i.e., good and bad news)
α	Parameter capturing the consumer's diminishing sensitivity away from the reference point

Appendix B: Preliminaries on Belief-Based Utility in Economics

Recent work in Economics suggest that agents realize utility from non-instrumental information. Consider the following examples by Dillenberger and Raymond (2020) and Ely et al. (2015) to better understand the notion of belief-based utility.

(a) An individual has a vacation upcoming in a few days. The weather on the day of the vacation – the outcome of interest – is uncertain and is realized only on the day of the vacation; the individual has preferences over the outcome (e.g., the individual prefers good weather over bad weather on the day of vacation). They monitor the weather forecast periodically and updates their belief (on the outcome) based on the forecast; the individual is a rational Bayesian. The individual may enjoy (resp., not enjoy) looking forward to the upcoming vacation if good (resp., bad) weather is more likely.

(b) An individual watches a sporting event/political debate. The identity of the eventual winner – the outcome of interest – is uncertain and is realized at the end of the game/debate. The individual has preferences over the outcome (e.g., the individual may have a favorite team/candidate and prefers their favorite team/candidate to win). They watch the game/debate; as it unfolds, the various events provide signals of the outcome (eventual winner). The individual periodically updates their belief on the outcome, based on these events. The individual may enjoy (resp., not enjoy) watching the game/debate if their favored team/player is more (resp., less) likely to win.[17]

In both these examples, observe that the agent realizes utility both from consumption and in the interim (before consumption), while no contingent action awaits. Below, we formally describe the canonical consumer-utility model from the recent literature in Economics, where the consumer realizes utility from consumption (material payoff/consumption utility) and *news* (news/belief-based utility) (Duraj and He 2019, Dillenberger and Raymond 2020).

B.1. Canonical Consumer-Utility Model: Consumption and Belief-Based Utilities

The model described below is a discrete-time model. Consider an environment that consists of one agent and nature. Time is discrete, consisting of $T \in \mathbb{I}^+$ periods and indexed by $t \in [T]$. At the start of period 1, nature chooses a state among a finite number of states $\Omega = \{\omega_1, \omega_2, \ldots, \omega_N\}$ from a distribution $\boldsymbol{\pi}_0 = \{\pi_0(\omega_i)\}_{i=1}^N$, but does not inform the agent. Let $\omega \in \Omega$ denote a generic state; $\boldsymbol{\pi}_0$ is common knowledge. The agent does not have any action. Consumption occurs at the end of period T, when the agent realizes a (state-dependent) consumption utility v_ω; consumption does not occur in any other periods. Without loss of generality, we assume that $\pi_0(\omega) > 0$ for all $\omega \in \Omega$, and $v_{\omega_1} < v_{\omega_2} < \ldots < v_{\omega_N}$. Therefore, ω_1 (resp., ω_N) is the worst (resp., best) state for the agent. Let $\boldsymbol{\Pi} = \Delta(\Omega)$ denote the set of possible distributions on Ω; $\boldsymbol{\pi}_0 \in \boldsymbol{\Pi}$.

In each period $t \in [T]$, nature provides the agent with *news*, i.e, messages (or signals) about ω through an (exogenous) messaging device. Let $\mathscr{M}$ denote the set of messages and $\mathbf{M} = \Delta(\mathscr{M})$ denote the set of all

[17] Figure 1 of Ely et al. (2015) refers to win probability plots – the likelihood of a player's win – during a game of tennis. Such in-game win probability plots are commonplace across sports.

distributions over $\mathscr{M}$. The messaging device sends messages according to $\boldsymbol{\sigma} = \{\sigma_t\}_{t=1}^T$, where $\sigma_t(\cdot|h^{t-1}, \omega) \in \mathbf{M}$ is a distribution over messages in period t that depends on the state ω and the history $h^{t-1} \in H^{t-1} := (\mathscr{M})^{t-1}$, i.e., messages sent thus far; $\boldsymbol{\sigma}$ is common knowledge.

At the end of each period $t \in [T]$, the agent, a rational Bayesian, forms a posterior belief $\boldsymbol{\pi}_t$ about the state ω after the history h^t of messages; clearly $\boldsymbol{\pi}_t \in \boldsymbol{\Pi}$. This belief $\boldsymbol{\pi}_t$ is rational, and is calculated using $\boldsymbol{\pi}_0$ and the knowledge of the messaging device $(\mathscr{M}, \boldsymbol{\sigma})$. In period T, nature reveals the state (which it chose at the start of period 1), say $\tilde{\omega}$, to the agent. Then, the agent's belief is the degenerate distribution $\boldsymbol{\pi}_T(\tilde{\omega}) = 1$ and $\boldsymbol{\pi}_T(\omega) = 0$ for any $\omega \neq \tilde{\omega}$, $\omega \in \Omega$. We denote the degenerate belief by $\boldsymbol{\pi}_T = 1 \circ \tilde{\omega}$.

The agent derives utility based on the changes in their belief about the consumption in period T. Specifically, let $\boldsymbol{\mu} : \boldsymbol{\Pi} \times \boldsymbol{\Pi} \mapsto \Re$ denote a mapping from their new and old beliefs into the real numbers. They realize $\boldsymbol{\mu}(\boldsymbol{\pi}_t|\boldsymbol{\pi}_{t-1})$ at the end of period t, $t \in [T]$. Utility flow is undiscounted and $\boldsymbol{\mu}(\cdot|\cdot)$ is fixed across periods. The total expected utility of the agent is the sum of the consumption utility in period T and the belief-based utilities across all periods $t \in [T]$.

$$\begin{aligned}
\text{Total Expected Utility} &= \mathbb{E}_{\omega \sim \boldsymbol{\pi}_0}\left[\underbrace{v_\omega}_{\text{consumption utility in state } \omega} + \sum_{t \in [T]} \underbrace{\boldsymbol{\mu}(\boldsymbol{\pi}_t|\boldsymbol{\pi}_{t-1})}_{\text{belief-based utility in period } t} \right], \\
&= \underbrace{\mathbb{E}_{\omega \sim \boldsymbol{\pi}_0}[v_\omega]}_{\text{Expected Consumption Utility, } U_M} + \underbrace{\sum_{t \in [T]} \mathbb{E}_{\omega \sim \boldsymbol{\pi}_0}\left[\boldsymbol{\mu}\left(\boldsymbol{\pi}_t|\boldsymbol{\pi}_{t-1}\right)\right]}_{\text{Expected Belief-Based Utility, } U_B} .
\end{aligned} \tag{B.33}$$

The first term (the expected consumption utility, U_M) is independent of $(\mathscr{M}, \boldsymbol{\sigma})$. Hence, it is sufficient to consider only the second term (the expected belief-based utility, U_B) in our comparison of different information-provisioning strategies (of a firm to a consumer).

B.2. Models for $\boldsymbol{\mu}(\cdot|\cdot)$

Broadly, papers that study *news* utility adopt one of two belief-based utility models (Dillenberger and Raymond 2020):

(a) Anticipatory Utility Models (AUM), and

(b) Changing Beliefs Models (CBM).

In both these models, the total utility is additively separable in the belief-based utility and the consumption utility; the difference lies in the model for belief-based utility. Consider a piece of news, that leads to a change in the agent's beliefs on ω from $\boldsymbol{\pi}$ to $\boldsymbol{\pi}'$. The agent's belief-based utility from consuming this piece of news under the two models is as follows:

(a) Under AUM (Caplin and Leahy 2001), the agent's belief-based utility depends on their absolute levels of beliefs:

$$\boldsymbol{\mu}(\boldsymbol{\pi}'|\boldsymbol{\pi}) = \mu(\boldsymbol{\pi}'),$$

where $\mu : \boldsymbol{\Pi} \mapsto \Re$ maps beliefs on ω to the real line.

(b) Under CBM (Kőszegi and Rabin 2009), the agent's belief-based utility depends on the change in their beliefs on ω:

$$\boldsymbol{\mu}(\boldsymbol{\pi}'|\boldsymbol{\pi}) = \mu\left(F(\boldsymbol{\pi}') - F(\boldsymbol{\pi})\right),$$

where $F : \boldsymbol{\Pi} \mapsto \Re$. Some examples are:

- Duraj and He (2019) employ a mean-based changing beliefs model, where $F(\boldsymbol{\pi}) = \mathbb{E}_{\omega\sim\boldsymbol{\pi}}[v_\omega]$. Therefore,

$$\boldsymbol{\mu}(\boldsymbol{\pi}'|\boldsymbol{\pi}) = \mu\left(\mathbb{E}_{\omega\sim\boldsymbol{\pi}'}[v_\omega] - \mathbb{E}_{\omega\sim\boldsymbol{\pi}}[v_\omega]\right). \tag{B.34}$$

- Kőszegi and Rabin (2009) briefly discuss the above model, but predominantly focus on a percentile-based changing beliefs model, where

$$\boldsymbol{\mu}(\boldsymbol{\pi}'|\boldsymbol{\pi}) = \int_{p=0}^{1} \mu\left(G^{-1}_{\omega\sim\boldsymbol{\pi}'}(p) - G^{-1}_{\omega\sim\boldsymbol{\pi}}(p)\right) dp,$$

where $G_{\omega\sim\boldsymbol{\pi}}(\cdot)$ (resp., $G^{-1}_{\omega\sim\boldsymbol{\pi}}(p)$) denotes the C.D.F. (resp., p^{th}-percentile) of ω with distribution $\boldsymbol{\pi}$.

In line with recent research that models belief-based utility from consumption of *news* (Duraj and He 2019), and for analytical tractability, we adopt the mean-based changing beliefs model. Furthermore, $\mu(\cdot)$ is increasing, differentiable everywhere except possibly at 0, and $\mu(0) = 0$. Kőszegi and Rabin (2006) refer to the above model as the *reference-dependent universal gain-loss utility* model, where:

(a) the gain-loss utility is itself derived from consumption utility, and

(b) the *reference* is determined endogenously through the prior beliefs, i.e., rational expectations held in the recent past about outcomes.

Using (B.34) and (B.33), the agent's total expected utility can be written as:

$$\text{Total Expected Belief-Based Utility, } U_B = \sum_{t\in[T]} \mathbb{E}_{\omega\sim\boldsymbol{\pi}_0}\left[\mu\left(\mathbb{E}_{\omega\sim\boldsymbol{\pi}_t}[v_\omega] - \mathbb{E}_{\omega\sim\boldsymbol{\pi}_{t-1}}[v_\omega]\right)\right]. \tag{B.35}$$

Remark B.1. **(Consequentialist Model)** A *consequentialist* model is one where an agent does not incur any belief-based utility, i.e.,

$$\boldsymbol{\mu}(\boldsymbol{\pi}'|\boldsymbol{\pi}) = 0.$$

In the standard consequentialist model, all information-disclosure strategies are identical. □

Remark B.2. **(Suspense and Surprise)** Ely et al. (2015) consider a model where the agent's belief-based utility from consuming a piece of news depends on the amount of *suspense* or *surprise*. In the suspense (resp., surprise) model, the agent has a preference for suspense (resp., surprise). Formally, consider the agent's belief-based utility in period t, where their beliefs are $\boldsymbol{\pi}_t$. Suspense is the amount of variability in period-$(t+1)$ beliefs (e.g., standard deviation of $\boldsymbol{\pi}_{t+1}$), while surprise is the difference in beliefs from period-$(t-1)$ to period-t (e.g., the Euclidean distance between $\boldsymbol{\pi}_{t-1}$ and $\boldsymbol{\pi}_t$). Formally,

- Under the suspense model,

$$\boldsymbol{\mu}(\boldsymbol{\pi}_t) = \mu\left(\mathbb{E}\left[\sum_{i\in[N]}(\pi_{t+1}(\omega_i) - \pi_t(\omega_i))^2\right]\right),$$

  where $\pi_{t+1}(\omega_i)$ is random and $\mathbb{E}[\pi_{t+1}(\omega_i)] = \pi_t(\omega_i)$, due to the Martingale property of Bayesian updating.

- Under the surprise model, the agent has a preference for surprise:

$$\boldsymbol{\mu}(\boldsymbol{\pi}_t|\boldsymbol{\pi}_{t-1}) = \mu\left(||\boldsymbol{\pi}_t - \boldsymbol{\pi}_{t-1}||_2\right),$$
$$\text{where } ||\boldsymbol{\pi}' - \boldsymbol{\pi}||_2 = \sum_{i\in[N]}(\pi'(\omega_i) - \pi(\omega_i))^2 \text{ for any } \boldsymbol{\pi}, \boldsymbol{\pi}' \in \Pi.$$

Their focus is on the entertainment value of news (e.g., sporting events, movies), which is different from the value of news (e.g., delay estimate, ETA, etc.) to a consumer in our model. $\square$

B.3. Main Differences Between Our Work and Extant Literature in Economics

Unlike prior work in Economics, where the payoff-relevant variable is an exogenous state of nature (e.g., the weather in Example (a); the winner of the sporting event/debate in Example (b)), the payoff-relevant variable for a consumer in our model is the length of the horizon (i.e., the delay). Furthermore, in prior work, the lack of information provision (i.e., no news) during a period does not affect the consumer's beliefs about the payoff-relevant variable, and hence the agent does not realize any belief-based utility. However, in our model, the mere passage of time provides information (*bad* news) to the consumer about the realized delay. The sender's (firm) messaging strategies we analyze are commonly observed in service processes.

Appendix C: Our Focus on Loss Aversion and Diminishing Sensitivity: Justification

An agent's evaluation of information (news) about outcomes is intrinsically related to how they evaluate outcomes. We first provide justification that not only are loss aversion and diminishing sensitivity common behavioral issues, they are common in the specific context we study, i.e., consumers' attitudes towards waiting post-sales.

We begin with examples from the extant literature; subsequently we discuss examples from practice.

- In their seminal work, Kumar et al. (1997) study the impact of providing consumers a wait-time guarantee on their waiting experience via laboratory experiments. They find that if the time guarantee is met, then satisfaction is positive at the end of the wait. However, if it is violated, then the satisfaction is negative at the end of the wait. Their work highlights consumers' reference-dependent attitudes toward wait, i.e., consumers evaluate their waiting experience using their actual wait time relative to a reference.
- A number of papers study whether people treat time like money (Leclerc et al. 1995, Antonides et al. 2002, Krishnamurthy and Kumar 2002) using laboratory experiments. A consistent finding in these experimental papers is the asymmetric evaluation of perceived wait with respect to a reference point. Antonides et al. (2002) conclude that "the evaluation of negative outcomes with respect to a reference point is generally convex with respect to a reference point and relatively steep, whereas for positive outcomes it is concave and relatively flat" (Kahneman and Tversky 1979).
- In the transportation literature, researchers have investigated the value of headway time (the time difference between departures) in public transport services. For example, Hanssen and Larsen (2020) analyze how consumers willingness-to-pay depends on the headway time for ferry services in Norway. They find that the marginal reduction in headway time increases consumers willingness-to-pay, but the effect is diminishing. This is akin to our idea of diminishing sensitivity relative to a reference point.
- Within the recent OM literature, Yu et al. (2021) study reference-dependent behavior of customers empirically using data from call-centers of an Israeli bank. They find evidence for loss aversion (over time) among customers, and that their reference-dependent model performs better than a model with non-linear waiting costs. Yu et al. (2022) study the impact of wait-time information provision on customers in a major ride-hailing platform. Specifically, they analyze the impact of providing a biased estimate of the anticipated wait, and then correcting this biased estimate (similar to "cancelled news" in our context). They find evidence of loss aversion when the platform provides either an upward- or a downward-biased estimate.

Beyond these, we mention examples from practice that discuss consumers' reference-dependent attitudes towards wait.

- In the context of appointments for medical services (e.g., at urgent care centers), Experity (2022), a leading software and services company for on-demand healthcare, reports that patients perceive time spent waiting beyond the scheduled appointment time (a "loss") to be intolerable; the same patient is unlikely to weigh an early start, i.e., a shorter wait than anticipated (a "gain") identical to a loss. In such environments where patients are unlikely to balk, how uncertainty about wait is resolved over time is critical to the patients' evaluation of their waiting experience.

- In Disney, the common tactic of over-estimating wait times, to surprise customers with a shorter-than-promised wait time, also points to loss-aversion among consumers in their evaluation of wait times (Yu 2020, Experity 2022). As mentioned above, Yu et al. (2022) find that ride-hailing platforms also employ an identical tactic.

Loss-aversion and diminishing sensitivity are important behavioral issues from the standpoint of a researcher studying the impact of sharing real-time progress information. Our analysis shows that in the presence of loss-aversion, which is equivalent to belief fluctuation aversion, the firm prefers to not share real-time progress information. This is because sharing real-time progress information leads to frequent fluctuations in the consumer's beliefs. In the presence of diminishing sensitivity, we find that if good news (i.e., low delay) is more likely, then the firm benefits from sharing real-time progress information, due to the concavity on the positive side. If bad news (i.e., high delay) is more likely, then providing the bad news as a lump-sum is better, due to the convexity on the negative side.

Appendix D: Two Important Remarks

Remark D.1. **(Abandonment by a Participating Consumer at $t > 0$ is Irrational)** In the case of continuous distributions (Section 3.1), recall the assumption that X_i is IFR. Since the IFR property is closed under convolution (Lemma F.2), we have that the total delay D is IFR. Further, IFR implies DMRL (decreasing mean residual lifetime; Lemma F.2).

Consider OP. The consumer's expected delay at time t is $\overline{D}_t = t + \mathsf{MRL}_D(t)$. Therefore, their anticipated material payoff at time t is:

$$U_M \Big|_t = v - \mathsf{MRL}_D(t) - t.$$

However, the cost, $-t$, is sunk and does not affect their decision at time t. Since $v - \mathsf{MRL}_D(t)$ is increasing in t (since D has DMRL), it holds that

$$v - \underbrace{\mathsf{MRL}_D(0)}_{n\overline{x}} > 0 \implies v - \mathsf{MRL}_D(t) > 0.$$

The LHS of the first inequality above is the expected material payoff to a customer at the time of purchase; by rationality, we know that this quantity is positive and, thus, the first inequality is true. Therefore, the second inequality is also true. Therefore, abandonment by a participating consumer at time $t > 0$ is irrational under OP.

Consider CTI. The consumer's expected delay at time t, where $\sum_{j=0}^{i-1} X_i \le t < \sum_{j=0}^{i} X_i$ for some $i \in [n]$ is

$$\overline{D}_t = t + \mathsf{MRL}_X\left(t - \sum_{j=0}^{i-1} X_i\right) + (n-i)\overline{x}.$$

Using an identical argument as above, it holds that

$$v - n\overline{x} > 0 \implies v - \mathsf{MRL}_X\left(t - \sum_{j=0}^{i-1} X_i\right) + (n-i)\overline{x} > 0,$$

where the LHS holds for any participating consumer.

Now, consider the case of a two-point distribution for the task durations. Consider OP. First, observe that abandonment by a consumer is irrational at $t \in (t_i, t_{i+1})$, $i \in \{0\} \cup [n]$ since no information arises in this interval. Second, the binomial distribution is IFR and hence has DMRL (Chapter 6 of Lai and Xie (2006)). Using an identical argument as before, it holds that at $t = t_i$, abandonment is irrational. Consider CTI. Abandonment is irrational at $t \neq \sum_{j=0}^{i} X_j + x_L$ since no new information arises, while at time $t = \sum_{j=0}^{i} X_j + x_L$, an identical argument as above holds. ∎

Remark D.2. **(Choice of the Number of Tasks, n)** Throughout our analysis, we assume that the process comprises of n tasks, where n is fixed. Indeed, our main results – the comparison of the two strategies (OP and CTI) in their respective settings – depend on the firm's choice of n. Suppose the firm's process allows for full flexibility, i.e., it allows for the firm to choose any $n \in \mathbb{I}^+$ s.t. the task durations $X_1, X_2, \ldots, X_n$ satisfy the assumptions made in our analysis in their respective settings (i.e., the task durations are i.i.d and IFR in Sections 3 and 4, and the task durations are independent and IFR in Section 5). Then, we have the following:

- Consider Theorem 1, which states that in an n-task process with i.i.d. durations, $\mathsf{OP} \succ \mathsf{CTI}$ iff $y_{(n)} < ny_{(1)}$. Suppose that the task durations are drawn from an exponential, normal or a uniform distribution. Then, for *any* choice of n, we have that $\mathsf{OP} \succ \mathsf{CTI}$.
- Similarly, consider Theorem 2 (resp., Theorem H.1), which states that in an n-task process with i.i.d. task durations, under a two-point distribution (resp., a general discrete distribution), $\mathsf{OP} \succ \mathsf{CTI}$ iff $y_{(n)} < ny_{(1)}$. In the latter, suppose that the task durations are drawn from a geometric distribution. Then, for *any* choice of n, we have that $\mathsf{OP} \succ \mathsf{CTI}$.
- Under independent, but non-identical task durations, consider Theorem 7, which states that $\mathsf{OP} \succ \mathsf{CTI}$ iff $\sum_{i \in [n]} y_{(i)} < y_{[n]}$. Suppose that the task durations are drawn independently from non-identical exponential distributions. Then, for *any* choice n of the firm, we have that $\mathsf{OP} \succ \mathsf{CTI}$.

In particular, suppose that the distribution of the total delay D belongs to the *infinitely divisible* class of distributions.[18] Such processes allow for full flexibility in that for any choice $n \in \mathbb{I}^+$, one can identify $X_1, X_2, \ldots, X_n$ that are i.i.d. and $D =_d X_1 + X_2 + \ldots X_n$.

Suppose the firm does not have full flexibility in its choice of n. Then, it may not be possible for the firm to choose any value of $n \in \mathbb{I}^+$ (i.e., the firm may be restricted in its choice of n), or it may violate the assumptions in our analysis. That is, we do not know the comparison of OP and CTI for general distributions and values of n (beyond the distributions we consider). Furthermore, it may be possible that the result may change depending on the value of n. ■

[18] The distribution of a real-valued random variable, Y is *infinitely divisible* if, for each $n \in \mathbb{I}^+$, there exists a sequence of independent, identically distributed random variables $(Y_1, Y_2, \ldots, Y_n)$ such that $Y_1 + Y_2 + \ldots Y_n$ has the same distribution as Y. Examples of infinitely divisible distributions include the normal distribution, the Cauchy distribution, the Levy distribution, the gamma distribution, the Poisson distribution and the negative binomial distribution. Infinitely divisible distributions generalize the class of *stable* distributions. A real-valued random variable Y has a stable distribution if, for any $n \in \mathbb{I}^+$, and a sequence of i.i.d. random variables $(Y_1, Y_2, \ldots, Y_n)$ each with the same distribution of Y, the sum $Y_1 + Y_2 + \ldots Y_n$ has the same distribution of $a_n + b_n Y$ for some $a_n \in \Re$ and $b_n > 0$.

Appendix E: Proofs of Technical Results/Statements in the Paper

Proof of Theorem 1: From (6) and (8), we have that:

$$\mathsf{OP} \succ \mathsf{CTI} \Leftrightarrow U_B^{\mathsf{OP}} > U_B^{\mathsf{CTI}} \implies ny_{(1)} > y_{(n)}. \quad \square$$

Proof of Theorem 2: Recall that $U_B^{\mathsf{CTI}} = -np(1-p)\Delta_\rho \Delta_x$, while $U_B^{\mathsf{OP}} = -\left(\sum_{i=0}^n q_i \tau_i\right)\Delta_\rho\Delta_x$. It suffices to compare the terms $np(1-p)$ $(=ny_{(1)})$ and $\sum_{i=0}^n q_i\tau_i$ $(=y_{(n)})$. Thus, it suffices to show:

$$y_{(n)} < ny_{(1)}. \quad \square$$

Proof of Theorem 3: Recall the expressions of U_B^{CTI} and U_B^{OP}: Observe that both U_B^{CTI} and U_B^{OP} are linear in ρ_N. Therefore,

$$\begin{aligned} U_B^{\mathsf{CTI}} > U_B^{\mathsf{OP}} &\Leftrightarrow 2(1-p)\sqrt{p} - \rho_N\left(2p\sqrt{1-p}\right) > (1-p)\sqrt{p}\left(\sqrt{2}(1-p) + 2p\sqrt{\frac{1}{2-p}}\right) - \rho_N\left((2-p)\sqrt{\frac{2}{2-p} - 2p + p^2}\sqrt{2 - \frac{2}{2-p}}\right) \\ &\Leftrightarrow \rho_N < \overline{\rho}_N. \quad \square \end{aligned}$$

Proof of Theorem 4: We calculate $\frac{d\overline{\rho}_N}{dp}$ and show that it is negative for $p \in (0,1)$. For convenience, denote the expression for $\overline{\rho}_N$ in (21) as $\frac{u}{v}$. Then, $\frac{d\overline{\rho}_N}{dp} = \frac{u'v - v'u}{v^2}$. It suffices to show that the numerator in the RHS, $u'v - v'u < 0$. The numerator of $\frac{d\overline{\rho}_N}{dp}$ is the sum of three terms:

$$\begin{aligned} &-\underbrace{\left[\frac{p}{\sqrt{2}}\left((\sqrt{2}-1)\sqrt{2-p} + (\sqrt{2-p}-\sqrt{2})p\right)\left(2\left(\sqrt{2(2-p)} + \sqrt{1-p} - \sqrt{2(2-p)(1-p)}\right) + \sqrt{2}p\left(-4\sqrt{2-p} + 2\sqrt{2-p}p + \sqrt{(2-p)(1-p)}\right)\right)\right]}_{Ⓐ} \\ &+\underbrace{\left[(1-p)\left(2-\sqrt{2}+p\left(\sqrt{2}-\frac{2}{\sqrt{2-p}}\right)\right)\left(2\left(\sqrt{2}-\sqrt{(2-p)(1-p)}\right) - \sqrt{2}p\left(3-p-\sqrt{1-p}\right)\right)\right]}_{Ⓑ} \\ &-\underbrace{\left[((2-p)(1-p)p)\left(\sqrt{2}-\frac{2}{\sqrt{2-p}}-\frac{p}{(2-p)^{\frac{3}{2}}}\right)\left(2\left(\sqrt{2}-\sqrt{(2-p)(1-p)}\right) - \sqrt{2}p\left(3-p-\sqrt{1-p}\right)\right)\right]}_{Ⓒ}. \end{aligned}$$

We show that Ⓐ, Ⓒ are positive, while Ⓑ is negative; thus, the RHS is negative. Consider Ⓐ (resp., Ⓑ, Ⓒ), that comprises of the product of three terms. Below, we show that each of the terms are positive (resp., the first and second terms are positive, while the third term is negative, and the first term is positive, while the second and third term are negative).

Below, we consider Ⓐ, Ⓑ and Ⓒ and show that Ⓐ, Ⓒ > 0 while Ⓑ < 0.

- Ⓐ: Observe that term Ⓐ is the product of three terms. The first term $\frac{p}{\sqrt{2}} > 0$. The second term in Ⓐ is the product of the second term in Ⓑ and $\sqrt{1-\frac{p}{2}}$. Below, we show that the second term in Ⓑ is positive; thus, the second term in Ⓐ is positive. Consider the last term. Substitute $1-p = y^2$; hence, $p \in (0,1) \Rightarrow y \in (0,1)$. The last term can be written as:

  $$\sqrt{2}y\left(\sqrt{2} - \sqrt{1+y^2}\left(1+y^2-2y^3\right)\right).$$

  Since $y \in (0,1)$, we focus on the term inside the brackets, and show that this term is positive. To show that

  $$\sqrt{2} > \sqrt{1+y^2}\left(1+y^2-2y^3\right),$$

we show that the expression in the RHS increases and then decreases. To see this, consider the first derivative w.r.t y:

$$\frac{d}{dy}\left(\sqrt{1+y^2}\left(1+y^2-2y^3\right)\right)=\frac{y}{\sqrt{1+y^2}}\left(3-6y+3y^2-8y^3\right).$$

Since $\frac{y}{\sqrt{1+y^2}}>0$ for $y\in(0,1)$, we consider the cubic polynomial in the brackets in the RHS above. The discriminant of this cubic polynomial evaluates to -14688; hence, the cubic polynomial has exactly one real root. This root is:

$$\begin{aligned}y_R &= \frac{1}{24}\left(3-135\sqrt[3]{\frac{2}{3942+864\sqrt{34}}}+\sqrt[3]{\frac{3942+864\sqrt{34}}{2}}\right)\\ &\approx 0.471.\end{aligned}$$

Therefore, the first derivative is positive in $y\in(0,y_R)$ and is negative in $y\in(y_R,1)$. Thus, the RHS attains a maximum at $y=y_R$. The maximum value attained (at $y=y_R$) is ≈ 1.119. Thus, the inequality holds.

- Ⓑ: Observe that Ⓑ is the product of three terms. The first term $1-p>0$ for $p\in(0,1)$. The second term can be rewritten as follows:

$$\sqrt{2}\left((\sqrt{2}-1)-\underbrace{p\left(\frac{\sqrt{2}-\sqrt{2-p}}{\sqrt{2-p}}\right)}_{\text{increasing in } p}\right)$$

We focus on the term inside the brackets. Observe that $p\left(\frac{\sqrt{2}-\sqrt{2-p}}{\sqrt{2-p}}\right)$ is strictly increasing in p; besides, $p\left(\frac{\sqrt{2}-\sqrt{2-p}}{\sqrt{2-p}}\right)\Big|_{p=1}=\sqrt{2}-1$. Thus, the second term is positive. To show that the last term is negative, it suffices to show the following (by rearranging the last term):

$$\sqrt{2}>\sqrt{(2-p)(1-p)}+\frac{p}{\sqrt{2-p}}.$$

We show that the RHS is decreasing in p. The LHS $\sqrt{2}=\left(\sqrt{(2-p)(1-p)}+\frac{p}{\sqrt{2-p}}\right)\Big|_{p=0}$. Thus, the inequality holds. To see that the RHS is decreasing in p, consider the first derivative:

$$\frac{d}{dp}\left(\sqrt{(2-p)(1-p)}+\frac{p}{\sqrt{2-p}}\right)=\frac{1}{2\sqrt{2-p}}\left(2+\frac{p}{2-p}-\frac{3-2p}{\sqrt{1-p}}\right).$$

We show that the term inside the bracket in the RHS above is negative. By rearranging these terms, it suffices to show that

$$(4-p)\sqrt{1-p}<(3-2p)(2-p).$$

Let $\sqrt{1-p}=z\Rightarrow p=1-z^2$. To show the above inequality, it suffices to show that

$$2z^4-z^3+3z^2-3z+1>0 \text{ for } z\in(0,1).$$

The discriminant of the quartic polynomial above evaluates to 1940. Further, $8a_4a_3-3a_2^2$ evaluates to 45.[19] Thus, this quartic polynomial does not have any real roots; hence the quartic polynomial is always positive (in fact, for $z\in(0,1)$, the polynomial attains a minimum value of ≈ 0.246 at $z=y_R\approx 0.471$).

[19] We use a_k to denote the coefficient of x^k in a polynomial of degree K, $\sum_{k=0}^{K}a_kx^k$.

- Ⓒ: Observe that term Ⓒ is the product of three terms. The first term $(2-p)(1-p)p>0$ for $p\in(0,1)$. The last term is identical to the last term in Ⓑ which was shown to be negative. We show that the second term is negative. To see this, we rewrite the second term as:

$$-\frac{1}{\sqrt{2-p}}\left(2+\frac{p}{2-p}-\sqrt{2}\sqrt{2-p}\right). \tag{E.36}$$

We show that the term inside the bracket is positive. Let $2-p=z^2$; thus, $p\in(0,1)\Rightarrow z\in(1,\sqrt{2})$. It suffices to show that

$$\sqrt{2}z^3-z^2-2<0.$$

This cubic polynomial is increasing in z in the interval $(1,\sqrt{2})$: The first derivative is $3\sqrt{2}z\left(z-\frac{\sqrt{2}}{3}\right)>0$ for $z\in(0,1)$. Besides, $\left(\sqrt{2}z^3-z^2-2\right)\Big|_{z=\sqrt{2}}=0$. Thus, the inequality holds.

Thus, $\overline{\rho}_N$ is decreasing in p. $\square$

Proof of Theorem 5: Both U_B^{CTI} and U_B^{OP} are linear in ρ_N. Thus, using straightforward algebra, we have that $U_B^{\mathsf{CTI}}>U_B^{\mathsf{OP}}$ iff the following condition holds:

$$\begin{aligned}2\left((1-p)\sqrt[\alpha]{p}-\rho_N p\sqrt[\alpha]{1-p}\right) &> (1-p)^2\sqrt[\alpha]{2p}+(2p-p^2)\left(-\rho_N\sqrt[\alpha]{2\left(\frac{(1-p)^2}{2-p}\right)}\right)+\\ &\quad 2p(1-p)\sqrt[\alpha]{\frac{p}{2-p}}p^2\left(-\rho_N\sqrt[\alpha]{2\left(\frac{1-p}{2-p}\right)}\right)\\ \Longrightarrow \rho_N &< \overline{\rho}_N(\alpha). \quad \square\end{aligned}$$

Proof of Theorem 6: Using (12), (15) and (18), we have that

$$\begin{aligned}U_B^{\mathsf{OP}} &= \sum_{i=0}^{n}\left[q_i\sqrt[\alpha]{\Delta_x\tau_i}-\rho_N\left(\sqrt[\frac{\alpha}{\alpha-1}]{\sum_{j=i+1}^{n}q_j\sqrt[\alpha]{\Delta_x q_i\tau_i}}\right)\right]\\ U_B^{\mathsf{CTI}} &= n\left((1-p)\sqrt[\alpha]{p\Delta_x}-\rho_N p\sqrt[\alpha]{(1-p)\Delta_x}\right).\end{aligned}$$

Both U_B^{OP} and U_B^{CTI} are linear in ρ_N. Therefore, $\mathsf{CTI}\succ\mathsf{OP}$ iff the following holds: $\rho_N<\overline{\rho}_N(n,\alpha)$. $\square$

Proof of Theorem 7: From (24) and (25), it follows that

$$\mathsf{OP}\succ\mathsf{CTI}\Leftrightarrow y_{[n]}<\sum_{i\in[n]}y_{(i)}.$$

Proof of Theorem 8: From (28) and (29), we have:

$$\mathsf{OP}\succ\mathsf{CTI}\Leftrightarrow y_{[n]}<\sum_{i\in[n]}y_{(i)}.$$

Proof of Theorem 9: First, we show that U_B^{OP} and U_B^{CTI} are linear in ρ_N.

$$\begin{aligned}U_B^{\mathsf{OP}} &= q_1\mu(\delta_1-d_1)+q_2\left(\mu(\delta_1-\delta_2)+\mu(\delta_2-d_2)\right)+q_3\left(\mu(\delta_1-\delta_2)+\mu(\delta_2-\delta_3)+\mu(\delta_3-d_3)\right).\\ &= (1-p_1)(1-p_2)\sqrt{p_1+p_2}+(p_1+p_2-2p_1p_2)\sqrt{\frac{p_1p_2}{p_1+p+2-p_1p_2}}+\\ &\quad\rho_N\left(\sqrt{\frac{(1-p_1)(1-p_2)(p_1+p_2)}{p_1+p_2-p_1p_2}}(2p_1p_2-(p_1+p_2))+p_1p_2\left(\sqrt{\frac{(1-p_1)(1-p_2)(p_1+p_2)}{p_1+p_2-p_1p_2}}+\sqrt{\frac{p_1+p_2-2p_1p_2}{p_1+p_2-p_1p_2}}\right)\right)\end{aligned}$$

Next,

$$
\begin{aligned}
U_B^{\mathsf{CTI}} &= \sum_i (1-p_i)\mu(p_i\Delta_{x_i}) + p_i\mu(-(1-p_i)\Delta_{x_i}) \\
&= (\sqrt{p_1}(1-p_1) + \sqrt{p_2}(1-p_2)) + \rho_N\left(p_1\sqrt{1-p_1} + p_2\sqrt{1-p_2}\right).
\end{aligned}
$$

It follows that $\mathsf{CTI} \succ \mathsf{OP}$ iff $\rho_N < \overline{\rho}_N$ (where $\overline{\rho}_N$ is defined in (31)).

Proof of Theorem 10: The proof strategy is identical to the proof of Theorem 9.

$$
\begin{aligned}
U_B^{\mathsf{OP}} &= (1-p_1)(1-p_2)\sqrt[\alpha]{p_1+p_2} + (p_1+p_2-2p_1p_2)\sqrt[\alpha]{\frac{p_1p_2}{p_1+p+2-p_1p_2}} + \\
&\rho_N\left(\sqrt[\alpha]{\frac{(1-p_1)(1-p_2)(p_1+p_2)}{p_1+p_2-p_1p_2}}(2p_1p_2-(p_1+p_2)) + p_1p_2\left(\sqrt[\alpha]{\frac{(1-p_1)(1-p_2)(p_1+p_2)}{p_1+p_2-p_1p_2}} + \sqrt[\alpha]{\frac{p_1+p_2-2p_1p_2}{p_1+p_2-p_1p_2}}\right)\right) \\
U_B^{\mathsf{CTI}} &= (\sqrt[\alpha]{p_1}(1-p_1) + \sqrt[\alpha]{p_2}(1-p_2)) + \rho_N\left(p_1\sqrt[\alpha]{1-p_1} + p_2\sqrt[\alpha]{1-p_2}\right)
\end{aligned}
$$

Since U_B^{OP} and U_B^{CTI} are linear in ρ_N, it follows that $\mathsf{CTI} \succ \mathsf{OP}$ iff $\rho_N < \overline{\rho}_N$ (where $\overline{\rho}_N$ is defined in (31)).

Proof of Theorem 11: Under an n-task process, recall that $\mathcal{D} = \{d_1, d_2, \ldots, d_{|\mathcal{D}|}\}$.

$$
\begin{aligned}
U_B^{\mathsf{OP}} &= \left(\sum_{i\in|\mathcal{D}|} q_i\sqrt[\alpha]{\delta_i - d_i}\right) - \rho_N\left(\sum_{i=1}^{|\mathcal{D}|}\left((1-\sum_{j\in[i]} q_j)\sqrt[\alpha]{-(\delta_i-\delta_{i+1})}\right)\right) \\
U_B^{\mathsf{CTI}} &= \sum_i (1-p_i)\sqrt[\alpha]{p_i\Delta_{x_i}} - \rho_N\sum_i p_i\sqrt[\alpha]{(1-p_i)\Delta_{x_i}}.
\end{aligned}
$$

From the above, we have that $\mathsf{CTI} \succ \mathsf{OP}$ iff $\rho_N < \overline{\rho}_N$ (where $\overline{\rho}_N$ is defined in (32)).

Appendix F: Supporting Results

Consider a random variable D with support $[0,\infty)$, p.d.f $f(\cdot)$, and C.D.F. $F(\cdot)$. Suppose that $f(\cdot)\in\mathcal{C}^2$, $\mathbb{E}[D]$ is finite. Let $\mathsf{MRL}_D(t)$ for a random variable D denotes the mean residual lifetime at time t, i.e.,

$$\mathsf{MRL}_D(t)=\mathbb{E}[D-t|D>t].$$

Let D_t denote the random variable $D|D>t$, and $f_{D_t}(\cdot)$ (resp., $F_{D_t}(\cdot)$) its p.d.f (resp., C.D.F.). Recall the definition of $\overline{D}_t$: $\overline{D}_t=\mathbb{E}[D|D>t]=t+\mathsf{MRL}_D(t)$.

Lemma F.1. *The following hold:*

(a) For any $t_2\geq t_1$, $D_{t_2}\geq_{st} D_{t_1}$.

(b) $\overline{D}_t$ is increasing in t.

Proof: (a) Consider any $d\in[t_1,t_2]$: It is straightforward that $F_{D_{t_1}}(d)\geq 0=F_{D_{t_2}}(d)$. Now, consider $d\in[t_2,\infty)$:

$$F_{D_{t_1}}(d)-F_{D_{t_2}}(d)=\frac{(F(t_2)-F(t_1))\,(1-F(d))}{(1-F(t_2))\,(1-F(t_1))}>0.$$

Thus, it follows that for any $d\in[t_1,\infty)$, $F_{D_{t_1}}(d)\geq F_{D_{t_2}}(d)$. Thus, $D_{t_2}\geq_{st} D_{t_1}$.

(b) Using the property of stochastic ordering, it follows that $\mathbb{E}[D_{t_2}]\geq\mathbb{E}[D_{t_1}]$ for any $t_2\geq t_1$. Thus, $\overline{D}_{t_2}\geq\overline{D}_{t_1}$; hence, $\overline{D}_t$ is increasing in t. Alternatively, $\overline{D}_t$ can be written as follows:

$$\overline{D}_t=\mathbb{E}[D|D>t]=\frac{\int_t^\infty xf(x)dx}{1-F(t)}.$$

Therefore,

$$\overline{D}_t'=\frac{1}{(1-F(t))^2}\left(-tf(t)\int_t^\infty f(x)dx+f(t)\int_t^\infty xf(x)dx\right)=\frac{f(t)}{\overline{F}(t)^2}\left(\int_{x=t}^\infty (x-t)f(x)dx\right)>0.$$

□

We state some useful results from the literature.

Lemma F.2. *(a) Suppose D is IFR. Then, $\mathsf{MRL}_D(t)$ is decreasing in t.*

(b) Let $X_i, i\in[n]$ denote independent random variables. Suppose $X_i, i\in[n]$ is IFR. Then, the convolution $\sum_{i=1}^n X_i$ is also IFR.

The first part – *IFR implies DMRL* – is a well-known result (Lai and Xie 2006). The second part – *IFR is closed under convolution* – can be found in Ross et al. (2005), Lai and Xie (2006).

Let $X_i, i\in[n]$ denote a collection of n i.i.d random variables, where

$$X_i\sim f(\cdot),F(\cdot),X_i\in[0,\infty).$$

For any n, let $X^{(n)}=\sum_{i\in[n]}X_i$; thus, $D=_d X^{(n)}$; let $f^{(n)}$ (resp., $F^{(n)}(\cdot)$) denote the p.d.f (resp., C.D.F.) of D. Now, define

$$y_{(n)}=\mathbb{E}\left[\mathsf{MRL}_{X^{(n)}}(X^{(n)})\right]$$

Thus, $y_{(n)}$ (resp., $y_{(1)}$) denotes the expected mean residual lifetime at the time of failure for D (resp., X). Let $x_w^{(n)}={F^{(n)}}^{-1}(w)$ denote the w^{th} quantile of $X^{(n)}$.

LEMMA F.3. *The following hold:*

(a) $X^{(n)} \geq_{st} X^{(n-1)}$.

(b) $x_w^{(n)} \geq x_w^{(n-1)}$.

Proof: (a) Let $\tilde{0}$ denote the random variable 0 w.p. 1. Then, $X^{(n-1)} = \sum_{i=1}^{n-1} X_i + \tilde{0}$, while $X^{(n)} = \sum_{i=1}^{n} X_i$. Since $X_n \geq_{st} \tilde{0}$, the result follows from Theorem 1.2.17 of Müller and Stoyan (2002).

(b) Since $X^{(n)} \geq_{st} X^{(n-1)}$, we have that:

$$
\begin{aligned}
&\underbrace{\mathbb{P}[X^{n-1} \leq a]}_{F^{(n-1)}(a)} \geq \underbrace{\mathbb{P}[X^{(n)} \leq a]}_{F^{(n)}(a)} \text{ for all } a \in [0,\infty) \\
\implies &\underbrace{\mathbb{P}[X^{n-1} \leq x_w^{(n-1)}]}_{w} \geq \mathbb{P}[X^{(n)} \leq x_w^{(n-1)}] \text{ for some } w \in [0,1] \\
\implies &\mathbb{P}[X^{(n)} \leq x_w^{(n)}] \geq \mathbb{P}[X^{(n)} \leq x_w^{(n-1)}] \\
\implies &x_w^{(n)} \geq x_w^{(n-1)}.
\end{aligned}
$$

□

LEMMA F.4. *$y_{(n)}$ is increasing in n.*

Proof:

$$
\begin{aligned}
y_{(n)} &= \int_{t=0}^{\infty} \int_{z=t}^{\infty} \left(z \frac{f^{(n)}(z)}{\overline{F}^{(n)}(t)} dz \right) f^{(n)}(t) dt \\
&= \int_{z=0}^{\infty} \int_{t=0}^{z} \left(\frac{f^{(n)}(t)}{\overline{F}^{(n)}(t)} dt \right) z f^{(n)}(z) dz \\
&= \int_{z=0}^{\infty} -\ln(\overline{F}^{(n)}(z)) z f^{(n)}(z) dz \\
&= \int_{w=0}^{1} -\ln(w) {F^{(n)}}^{-1}(1-w) dw = \int_{w=0}^{1} \ln\left(\frac{1}{w}\right) {F^{(n)}}^{-1}(1-w) dw
\end{aligned}
$$

Thus,

$$
\delta_{(n)} = y_{(n)} - y_{(n-1)} = \int_{w=0}^{1} \ln\left(\frac{1}{w}\right) \underbrace{\left({F^{(n)}}^{-1}(1-w) - {F^{(n-1)}}^{-1}(1-w)\right)}_{x_{(1-w)}^{(n)} - x_{(1-w)}^{(n-1)}}
$$

The term inside the brackets corresponds to the difference between $x_{(1-w)}^{(n)}$ and $x_{(1-w)}^{(n-1)}$; using part (b) of Lemma F.3, this is non-negative. Thus, $\delta_{(n)} \geq 0$. □

LEMMA F.5. *$y_{(n)}$ is discrete concave in n if the following holds: $x_w^{(n)} (= {F^{(n)}}^{-1}(w))$ is discrete concave in n for all $w \in (0,1)$. This condition holds for the Normal distribution.*

Proof:

$$
\delta_{(n)} - \delta_{(n-1)} = \int_{0}^{1} \ln\left(\frac{1}{w}\right) \left(\left(\underbrace{{F^{(n)}}^{-1}(1-w) - {F^{(n-1)}}^{-1}(1-w)}_{x_{(1-w)}^{(n)} - x_{(1-w)}^{(n-1)}} \right) - \left(\underbrace{{F^{(n-1)}}^{-1}(1-w) - {F^{(n-2)}}^{-1}(1-w)}_{x_{(1-w)}^{(n-1)} - x_{(1-w)}^{(n-2)}} \right) \right).
$$

Since $x_w^{(n)}$ is increasing and concave in n at all $w \in [0,1]$, the term inside the bracket is non-negative for all w. It follows that $\delta_{(n)} \leq \delta_{(n-1)}$.

Next, consider $X_i \sim \mathcal{N}(\mu, \sigma^2)$, with $\mu > 0$. For any $w \in [0,1]$, let $z_{(w)}$ denote the w^{th} quantile of the standard normal distribution, i.e.,

$$z_{(w)} = \Phi^{-1}(w),$$

where $\Phi(\cdot)$ denotes the C.D.F. of the standard normal. Thus,

$$F^{(n)^{-1}}(1-w) = n\mu + \sqrt{n}\sigma z_{(1-w)}.$$

Therefore, $F^{(n)^{-1}}(1-w) - F^{(n-1)^{-1}}(1-w) = \mu + \left(\sqrt{n} - \sqrt{n-1}\right)\sigma z_{(1-w)}$. Since $(\sqrt{n} - \sqrt{n-1})$ is (strictly) decreasing in n, it holds that $F^{(n)^{-1}}(1-w)$ is (strictly) discrete concave in n for all $w \in (0,1)$ $\square$

Lemma F.6. (9) *holds for the exponential and normal distribution.*

Proof: First, it suffices to show the result for the exponential and normal distributions with means 1.

Consider $X_i \sim \mathsf{Exp}(1)$. Using the memorylessness property of exponential random variables, we have that $y_{(1)} = 1$. Since the sum of exponential distributions follows an Erlang distribution, which is DMRL, we have that $y_{(n)} < \mathsf{MRL}_D(0) = n$. Thus, (9) holds for the exponential distribution.

Consider $X_i \sim \mathcal{N}(1, \sigma^2)$. The proof follows in a straightforward manner from Lemma F.5. $\square$

Lemma F.7. *Consider $X_i \sim p \circ x_H + (1-p) \circ x_L$.* (17) *holds for $n = 2$ and 3.*

Proof: We calculate the difference $y_{(n)} - ny_{(1)}$ for $n = 2$ and 3 and show that the difference is negative for $p \in (0,1)$. Further, without l.o.g, we normalize $\Delta_x = 1$. For any n and $\Delta_x = 1$, from (13) we have the following:

$$y_{(n)} = \sum_{i=0}^{n} \sum_{j=i}^{n} \frac{q_j q_i}{\sum_{k=i}^{n} q_k}(j-i).$$

- $n = 2$: Using straightforward algebra,

$$ny_{(1)} = 2p(1-p) \text{ and } y_{(2)} = \frac{2(1-p)p(p^2 + 2(1-p))}{2-p}$$

  Therefore,

$$y_{(2)} - ny_{(1)} = -\frac{2(1-p)^2p^2}{2-p} < 0.$$

- $n = 3$: Using straightforward algebra,

$$ny_{(1)} = 3p(1-p) \text{ and } y_{(3)} = \frac{3(1-p)p}{(3-2p)(3-(3-p)p)}\left(9 - 24p + 33p^2 - 27p^4 + 12p^5 - 2p^6\right).$$

  Therefore,

$$y_{(3)} - ny_{(1)} = -\left(\frac{3(3-p)(1-p)^2p^2}{(3-p(3-p))(3-2p)}\right)\left(3 - 4p + 2p^2\right) < 0.$$

  The first term is strictly positive in $p \in (0,1)$. The second term is decreasing in $p \in (0,1)$; at $p = 1$, the second term is 0. Thus, the second term is also positive in $p \in (0,1)$. $\square$

Remark F.1. In Lemma F.7, we show that $y_{(n)} < ny_{(1)}$ for $n = 2, 3$. For higher values of n, the expressions for $y_{(n)}$ are cumbersome. The details of the numerical verification of $y_{(n)} < ny_{(1)}$ for higher values of n using Mathematica are available upon request.

F.1. An Upper Bound for $y_{(n)}$ Under the Discrete (Two-Point) Distribution

Consider a two-point task distribution as follows:

$$X_i \sim p \circ x_H + (1-p) \circ x_L,$$

where $x_H > x_L > 0$. Define $\Delta_x = x_H - x_L$; w.l.o.g., $\Delta_x = 1$. Let $D = \sum_{i\in[n]} X_i$ and $t_i = ix_H + (n-i)x_L$, $i \in \{0,1,\dots,n\}$. Therefore,

$$q_i = \mathbb{P}[D = ix_H + (n-i)x_L] = \binom{n}{i} p^i (1-p)^{n-i} \text{ for } i \in \{0,1,\dots,n\}.$$

Recall the definition of $y_{(n)}$:

$$\begin{aligned} y_{(n)} = \mathbb{E}[\mathsf{MRL}_D(D^-)] &= \sum_{i=0}^{n} q_i \tau_i, \text{ where } \tau_i = \mathbb{E}_{I\sim \mathsf{Bin}(n,p)}[I-i|I\geq i] \\ &= \sum_{i=0}^{n}\sum_{j=i}^{n} \frac{q_i q_j}{\sum_{j'=i}^{n} q_{j'}}(j-i). \end{aligned}$$

Let $I, \tilde{I}$ denote independent binomial r.v.'s with parameters n, p. Let $Z_i, \tilde{Z}_i$, $i \in \{1,2,\dots\}$ denote independent Bernoulli r.v.'s with parameter p.

Define $w_{(n)}$ as follows:

$$\begin{aligned} w_{(n)} &= \mathbb{E}\left[\max\left\{I,\tilde{I}\right\} - \min\left\{I,\tilde{I}\right\}\right] \\ &= \mathbb{E}\left[\max\left\{\sum_{i\in[n]} X_i, \sum_{i\in[n]} \tilde{X}_i\right\}\right] - \mathbb{E}\left[\min\left\{\sum_{i\in[n]} X_i, \sum_{i\in[n]} \tilde{X}_i\right\}\right]. \end{aligned}$$

With a mild abuse of notation, let $w_{(n)}^{(a)}$, $a \geq 0$, denote the following:

$$w_{(n)}^{(a)} = \mathbb{E}\left[\max\left\{\sum_{i\in[n]} X_i + a, \sum_{i\in[n]} \tilde{X}_i\right\}\right] - \mathbb{E}\left[\min\left\{\sum_{i\in[n]} X_i + a, \sum_{i\in[n]} \tilde{X}_i\right\}\right] \tag{F.37}$$

For convenience, $w_{(n)}^{(0)} = w_{(n)}$. First, the following result establishes our main recursive equation to compute $w_{(n)}^{(a)}$.

Lemma F.8. *Fix $a \geq 0$. The following holds:*

$$w_{(n+1)}^{(a)} = \left(p^2 + (1-p)^2\right) w_{(n)}^{(a)} + p(1-p) w_{(n)}^{(a+1)} + p(1-p) w_{(n)}^{|a-1|}.$$

Consequently,

$$w_{(n+1)} = \left(p^2 + (1-p)^2\right) w_{(n)} + 2p(1-p) w_{(n)}^{(1)}.$$

Proof: Consider (F.37).

$$\begin{aligned} w_{(n+1)}^{(a)} &= \mathbb{E}\left[\max\left\{\sum_{i\in[n+1]} X_i + a, \sum_{i\in[n+1]} \tilde{X}_i\right\} - \min\left\{\sum_{i\in[n+1]} X_i + a, \sum_{i\in[n+1]} \tilde{X}_i\right\}\right] \\ &= \sum_{x_{n+1},\tilde{x}_{n+1}\in\{0,1\}^2} \mathbb{P}\left[X_{n+1} = x_{n+1}, \tilde{X}_{n+1} = \tilde{x}_{n+1}\right] \times \\ &\quad \mathbb{E}\left[\max\left\{\sum_{i\in[n]} X_i + x_{n+1} + a, \sum_{i\in[n]} \tilde{X}_i + \tilde{x}_{n+1}\right\}\right] - \mathbb{E}\left[\min\left\{\sum_{i\in[n]} X_i + x_{n+1} + a, \sum_{i\in[n]} \tilde{X}_i + \tilde{x}_{n+1}\right\}\right] \\ &= (p^2 + (1-p)^2) w_{(n)}^{(a)} + p(1-p) w_{(n)}^{(a+1)} + p(1-p) w_{(n)}^{|a-1|}. \end{aligned}$$

Substituting $a = 0$, we have the second part of the result.

LEMMA F.9. *Fix $a \geq 0$. The following statements hold:*

(a) $w_{(n)}^{(a)}$ *is increasing in a for fixed n.*

(b) $w_{(n)}^{(a)} = a$ *for $a \geq n$.*

(c) $w_{(n)}^{(a)} - w_{(n)} \leq a$ *for any a, n.*

Proof: To show part (a), observe that:

$$\begin{aligned}
\frac{dw_{(n)}^{(a)}}{da} &= \mathbb{P}\left[\sum_{i\in[n]} X_i + a \geq \sum_{i\in[n]} \tilde{X}_i\right] - \mathbb{P}\left[\sum_{i\in[n]} X_i + a < \sum_{i\in[n]} \tilde{X}_i\right] \\
&= 2\left(\underbrace{\mathbb{P}\left[\sum_{i\in[n]} U_i + a \geq 0\right]}_{\text{increasing in } a} - \frac{1}{2}\right)
\end{aligned}$$

where

$$U_i =_d X_i - \tilde{X}_i = \begin{cases} 1, & \text{w.p. } p(1-p); \\ 0, & \text{w.p. } p^2 + (1-p)^2; \\ -1, & \text{w.p. } p(1-p). \end{cases}$$

The RHS of $\frac{dw_{(n)}^{(a)}}{da}$ is increasing in a.

To show part (b), consider the RHS of (F.37) for $a \geq n$:

$$\begin{aligned}
w_{(n)}^{(a)} &= a + \mathbb{E}\left[\sum_i X_i\right] - \mathbb{E}\left[\sum_i \tilde{X}_i\right] \\
&= a.
\end{aligned}$$

To show part (c), consider $0 \leq a < n$.

$$\begin{aligned}
w_{(n)}^{(a)} &= \mathbb{P}\left(\sum_{i\in[n]} X_i \geq \sum_{i\in[n]} \tilde{X}_i\right) \times \mathbb{E}\left[a + \sum_{i\in[n]} X_i - \sum_{i\in[n]} \tilde{X}_i | \sum_{i\in[n]} X_i \geq \sum_{i\in[n]} \tilde{X}_i\right] \\
&\quad + \mathbb{P}\left(\sum_{i\in[n]} X_i < \sum_{i\in[n]} \tilde{X}_i, \sum_{i\in[n]} X_i + a \geq \sum_{i\in[n]} \tilde{X}_i\right) \times \\
&\quad \mathbb{E}\left[a + \sum_{i\in[n]} X_i - \sum_{i\in[n]} \tilde{X}_i | \sum_{i\in[n]} X_i < \sum_{i\in[n]} \tilde{X}_i, \sum_{i\in[n]} X_i + a \geq \sum_{i\in[n]} \tilde{X}_i\right] \\
&\quad + \mathbb{P}\left[\sum_{i\in[n]} X_i + a < \sum_{i\in[n]} \tilde{X}_i\right] \times \mathbb{E}\left[\sum_{i\in[n]} \tilde{X}_i - \sum_{i\in[n]} X_i | \sum_{i\in[n]} X_i + a < \sum_{i\in[n]} \tilde{X}_i\right] \\
&= a\left(\mathbb{P}\left[\sum_{i\in[n]} X_i \geq \sum_{i\in[n]} \tilde{X}_i\right] + \mathbb{P}\left[\sum_{i\in[n]} X_i < \sum_{i\in[n]} \tilde{X}_i, \sum_{i\in[n]} X_i + a \geq \sum_{i\in[n]} \tilde{X}_i\right]\right) + w_{(n)}. \\
\implies w_{(n)}^{(a)} - w_{(n)} &< a.
\end{aligned}$$

We now establish our main result.

Lemma F.10. *The following holds:*

$$w_{(n)} < n w_{(1)}.$$

where $w_{(1)} = 2p(1-p)$.

Proof: We rewrite the recursive equation above as:

$$\begin{aligned}
w_{(n+1)} =& w_{(n)} + 2p(1-p)\left(w^{(1)}_{(n)} - w_{(n)}\right) \\
=& w_{(n)} + w_{(1)}\left(w^{(1)}_{(n)} - w_{(n)}\right) && \text{(since } w_{(1)} = 2p(1-p)\text{)} \\
<& w_{(n)} + w_{(1)} && \text{(from part (c) of Lemma above, } w^{(1)}_{(n)} - w_{(n)} < 1\text{)} \\
<& (n+1)w_{(1)} && \text{(induction on } n\text{)}
\end{aligned}$$

Lemma F.11. *For any $n > 1$, we have:*

$$\frac{w_{(n)}}{2} < y_{(n)} < \frac{n y_{(1)}}{p^n}$$

Proof: We have the following:

$$\begin{aligned}
\frac{w_{(n)}}{2} &= \sum_{i=0}^{n}\sum_{j=i}^{n} q_i q_j (j-i) \\
y_{(n)} &= \sum_{i=0}^{n}\sum_{j=i}^{n} q_i \left(\frac{q_j}{\sum_{j' \geq i} q_{j'}}\right)(j-i).
\end{aligned}$$

From the above, it is straightforward that $y_{(n)} > \frac{w_{(n)}}{2}$. Further, consider the RHS of $y_{(n)}$ above. We have:

$$y_{(n)} < \frac{1}{p^n}\left(\sum_{i=0}^{n}\sum_{j=i}^{n} q_i q_j (j-i)\right) = \frac{1}{p^n}\frac{w_{(n)}}{2} = \frac{1}{p^n} n y_{(1)}.$$

Combining the two inequalities above, we have the required result.

Appendix G: Expected Mean Residual Lifetime at the Time of Failure for the $\mathsf{UniformSum}(n,\{0,2\})$ Distribution

Below, we show that (9) holds for the uniform distribution with positive support. It suffices to show this result for $X_i =_d \hat{X}_i \sim \mathsf{Uniform}[0,2]$. For $X_i \sim \mathsf{Uniform}[a,b]$, where $0 \le a < b$, we have that:

$$\begin{aligned}\mathbb{E}[\mathsf{MRL}_X(X)] &= \left(\frac{b-a}{2}\right)\mathbb{E}[\mathsf{MRL}_{\hat{X}}(\hat{X})]\\ \mathbb{E}[\mathsf{MRL}_D(D)] &= \left(\frac{b-a}{2}\right)\mathbb{E}[\mathsf{MRL}_{\hat{D}}(\hat{D})]\end{aligned}$$

where $D = \sum_{i\in[n]} X_i$ and $\hat{D} = \sum_{i\in[n]} \hat{X}_i$. It is straightforward that $\hat{D}$ follows the $\mathsf{UniformSum}(n,\{0,2\})$ distribution (also called $\mathsf{IrwinHall}(n,\{0,2\})$ distribution).

Let

$$y_{(1)} = \mathbb{E}[\mathsf{MRL}_{\hat{X}}(\hat{X})] \text{ and } y_{(n)} = \mathbb{E}[\mathsf{MRL}_{\hat{D}}(\hat{D})].$$

For $n=2$, we have the following:

- Since $\hat{X} \sim U[0,2]$, we have that $\mathsf{MRL}_{\hat{X}}(t) = 1 - \frac{t}{2}$. Therefore, $y_{(1)} = \frac{1}{2}$.
- Next, since $\hat{D} \sim \mathsf{UniformSum}(n,\{0,2\})$, we have that

$$\mathsf{MRL}_{\hat{D}}(t) = -t + \begin{cases} \frac{2}{3}(2+t), & \text{if } t \in [2,4];\\ \frac{2}{3}\left(\frac{24-t^3}{8-t^2}\right), & \text{if } t < 2.\end{cases}$$

  Therefore, $y_{(2)} = \mathbb{E}[\mathsf{MRL}_{\hat{D}}(\hat{D})] \approx 0.729$.

Combining these two, we have:

$$y_{(2)} < 2y_{(1)}.$$

For higher values of n ($n \le 20$), we provide the following table for $y_{(n)}$. It is straightforward to verify that $y_{(n)} < ny_{(1)}$.

n	y_n	n	y_n	n	y_n	n	y_n
1	0.5	6	1.27443	11	1.72755	16	2.08429
2	0.729093	7	1.37707	12	1.80456	17	2.14854
3	0.897992	8	1.47255	13	1.8784	18	2.21091
4	1.03896	9	1.56219	14	1.94945	19	2.27133
5	1.16272	10	1.64695	15	2.018	20	2.33039

Table G.1 Mean Residual Life at the Time of Failure for an $\mathsf{UniformSum}(n,\{0,2\})$ **Random Variable**

Appendix H: Analysis under General Discrete Distributions

Consider the setting where X_i's are i.i.d. Consider the following discrete distribution for the task durations with support $\{0,1,2,\dots\}$ and the following p.m.f:

$$X_i \sim \sum_{m=0}^{\infty} p_m \circ m; \quad \mathbb{E}[X_i] = \overline{x}.$$

We assume that X_i is IFR. The total delay $D = \sum_{i\in[n]} X_i$ is as follows:

$$D = \sum_{i\in[n]} X_i \sim \sum_{m=0}^{\infty} r_m \circ m; \quad \mathbb{E}[D] = n\overline{x}.$$

where $r_m = \sum_{i_1\in\{0\}\cup[m]} \sum_{i_2\in\{0\}\cup[m-i_1]} \cdots \sum_{i_n\in\{0\}\cup[m-\sum_{j\in[n-1]} i_j]} \left(\prod_{k=1}^{n} p_{i_k}\right)$.

- Consider OP: Consider a realization of D, say D. Let δ_t denote the consumer's mean belief on D at any time $t \le D$.

$$\overline{D}_t = \delta_t \triangleq \mathbb{E}[D|D \ge t] = t + \mathsf{MRL}_D(t^-).$$

For any $t_2 > t_1 \ge 0$, it is straightforward to verify that $(D|D \ge t_2) \ge_{st} (D|D \ge t_1)$; therefore, δ_t is increasing in t. OP resolves uncertainty on D as follows:

$$n\overline{x} - D = \underbrace{(\delta_0 - \delta_1)}_{\textit{bad news} \text{ at } t=0^+} + \underbrace{(\delta_1 - \delta_2)}_{\textit{bad news} \text{ at } t=1^+} + \dots + \underbrace{(\delta_{D-1} - \delta_D)}_{\textit{bad news} \text{ at } t=(D-1)^+} + \underbrace{(\delta_D - D)}_{\textit{good news} \text{ at } t=D^+}$$

That is, under OP, the consumer receives bad news in $t \in \{0,1,2,\dots,D-1\}$ (since δ_t is increasing in t) and good news at $t = D$. Now, define the expected mean residual life at the time of failure for D as follows:

$$y_{(n)} = \mathbb{E}[\mathsf{MRL}_D(D^-)].$$

The consumer's expected belief-based utility under OP under a piecewise-linear utility model is:

$$U_B^{\mathsf{OP}} = -\Delta_\rho y_{(n)}. \tag{H.38}$$

- Consider CTI: Consider realizations, say $X_1, X_2, \dots, X_n$. Let

$$\tau_x = \mathbb{E}[X_i|X_i \ge x] = x + \mathsf{MRL}_X(x^-).$$

As before, τ_x is increasing in x. At any time $t \in [\sum_{j=0}^{i-1} X_j, \sum_{j=0}^{i} X_j)$, the consumer's belief on D at time t is as follows:

$$\overline{D}_t = t + \mathsf{MRL}_X\left(t - \sum_{j=0}^{i-1} X_j\right) + (n-i)\overline{x}.$$

CTI resolves uncertainty on D as follows:

$$n\overline{x} - D = \sum_{i=1}^{n}\left[\underbrace{(\tau_0 - \tau_1)}_{\textit{bad news} \text{ at } t=\sum_{j=0}^{i-1} X_j+0^+} + \underbrace{(\tau_1 - \tau_2)}_{\textit{bad news} \text{ at } t=\sum_{j=0}^{i-1} X_j+1^+} + \dots + \underbrace{(\tau_{X_i-1} - \tau_{X_i})}_{\textit{bad news} \text{ at } t=\sum_{j=0}^{i-1} X_j+(X_i-1)^+} + \underbrace{(\tau_{X_i} - X_i)}_{\textit{good news} \text{ at } t=\sum_{j=0}^{i-1} X_j+X_i^+}\right].$$

That is, CTI resolves uncertainty task-by-task. Hence, in the interval $t \in [\sum_{j=0}^{i-1} X_j, \sum_{j=0}^{i} X_j)$ the consumer receives $(X_i - 1)$ pieces of bad news (since τ_x is increasing in x). At $t = \sum_{j=0}^{i} X_j$, the consumer receives good news. Define

$$y_{(1)} = \mathbb{E}[\mathsf{MRL}_X(X^-)].$$

The consumer's (expected) belief-based utility under CTI under a piecewise linear utility model can be written as:

$$U_B^{\mathsf{CTI}} = -\Delta_\rho n y_{(1)} \tag{H.39}$$

Therefore, we have the following result:

THEOREM H.1. *Suppose $\mu(\cdot) \in \mathscr{U}^{\mathsf{PL}}$. $\mathsf{OP} \succ \mathsf{CTI}$ iff the following condition holds:*

$$y_{(n)} < n y_{(1)}.$$

The above condition holds under the geometric distribution for task durations.

Proof: The first part follows from (H.38) and (H.39).

For the second part, assume $X_i \sim \mathsf{Geometric}(p)$; thus, $\overline{x} = \frac{1}{p}$. The mean residual lifetime at any time $t \in \mathbb{I}^+$ of the geometric distribution is a constant and is equal to $\frac{1}{p}$, i.e, $\mathsf{MRL}_X(t^-) = \frac{1}{p}$. Since $X_i \sim \mathsf{Geometric}(p)$, it follows that $D \sim \mathsf{NegativeBinomial}(n,p)$. The negative binomial distribution is IFR and hence DMRL (Lai and Xie 2006). Hence,

$$y_{(n)} = \mathbb{E}[\mathsf{MRL}_D(D^-)] < \mathsf{MRL}_D(0) = \frac{n}{p} = n y_{(1)},$$

which is the required result. $\square$

Further, analogous to Remark D.1, it is straightforward to verify that under OP and CTI, abandonment by a participating consumer at $t > 0$ is irrational.

Now, consider the case where X_i's are independent, but not i.i.d. Let $\{0,1,2,\ldots\}$ be the support of X_i's, and the following p.m.f. for X_i:

$$X_i \sim \sum_{m=0}^{\infty} p_{im} \circ m; \quad \mathbb{E}[X_i] = \overline{x}_i.$$

The p.m.f. of the total delay, $D = \sum_i X_i$ be denoted by the following:

$$D \sim \sum_{m=0}^{\infty} q_m \circ m; \quad \mathbb{E}[D] = \sum_i \overline{x}_i.$$

Let

$$\delta_t = \mathbb{E}[D | D \geq t] = t + \mathsf{MRL}_D(t^-).$$

Indeed, $\delta_0 = \mathbb{E}[D]$. Under OP, the total stock of news is resolved as follows:

$$\sum_i \overline{x}_i - D = (\delta_0 - \delta_1) + (\delta_1 - \delta_2) + \ldots + (\delta_D - D).$$

Let

$$\tau_{it} = \mathbb{E}[X_i | X_i \geq t] = t + \mathsf{MRL}_{X_i}(t^-).$$

Indeed, $\tau_{i0} = \overline{x}_i$. Under CTI, the total stock of news is resolved as follows:

$$\sum_i \overline{x}_i - D = \sum_i \left[(\tau_{i0} - \tau_{i1}) + (\tau_{i1} - \tau_{i2}) + \ldots + (\tau_{iX_i} - X_i) \right].$$

Let $y_{[n]}$ and $y_{(i)}$ denote the following:

$$\begin{aligned} y_{[n]} &= \mathbb{E}[\mathsf{MRL}_D(D)] \\ y_{(i)} &= \mathbb{E}[\mathsf{MRL}_{X_i}(X_i)] \end{aligned}$$

Therefore, the consumer's (expected) belief based utility under a piecewise linear utility model can be written as follows:

$$U_B^{\mathsf{OP}} = -\Delta_\rho y_{[n]} \tag{H.40}$$

$$U_B^{\mathsf{CTI}} = -\Delta_\rho \sum_i y_{(i)}. \tag{H.41}$$

The following result compares the two strategies.

THEOREM H.2. *Suppose $\mu(\cdot) \in \mathscr{U}^{\mathsf{PL}}$. $\mathsf{OP} \succ \mathsf{CTI}$ iff the following condition holds.*

$$y_{[n]} < \sum_i y_{(i)}.$$

The above condition is satisfied by the geometric distribution for task durations.

Proof: The first part of the result follows directly from (H.40) and (H.41). For the second part, let $X_i \sim_\perp \mathsf{Geometric}(p_i)$. The proof is similar to the proof of Theorem H.1. Since the sum of independent IFR random variables is IFR, and hence DMRL, we have that

$$\begin{aligned} \underbrace{y_{[n]}}_{\mathbb{E}[\mathsf{MRL}_D(D)]} &< \sum_i \underbrace{\overline{x}_i}_{\mathsf{MRL}_D(0)} \\ &= \sum_i \frac{1}{p_i} = \sum_i y_{(i)}. \end{aligned}$$

Appendix I: Calculation of $y_{(n)}$ for Some IFR Distributions

Recall that Theorem 1 (comparison of OP and CTI under i.i.d. task durations, in the case of continuous distributions) states the following. $\mathsf{OP} \succ \mathsf{CTI}$ iff the following condition holds:

$$y_{(n)} < n y_{(1)},$$

where $y_{(n)} = \mathbb{E}[\mathsf{MRL}_D(D)]$ and $y_{(1)} = \mathbb{E}[\mathsf{MRL}_X(X)]$, i.e., $y_{(n)}$ (resp., $y_{(1)}$) are the expected mean residual life at the time of failure for the n-fold convolution of X (resp., for X).

We now offer numerical support for the validity of the inequality above in Theorem 1 for three other well-known IFR distributions – Gamma, Lognormal, and Weibull – that are commonly used in the stochastic scheduling literature for modeling task durations (Pinedo 2016). We calculate the quantity $y_{(i)}$, $i \in \{1, 2, \ldots, 10\}$ via a numerical simulation. We normalize the mean of the task durations to 1 without loss of generality. Note that the quantities calculated above are the sample means of our large numerical simulation, and have an associated standard error.

Through our numerical simulation, we find that each of these three distributions – Gamma, Lognormal and Weibull – satisfy the inequality $y_{(n)} < n y_{(1)}$. While this certainly does not constitute an analytical proof, it helps us offer more support for the dominance of the OP strategy over the CTI strategy for IFR task distributions.

I.1. Simulation Procedure

For a random variable, say X with p.d.f. (resp., C.D.F.) $f(\cdot)$ (resp., $F(\cdot)$), the expected mean residual life at the time of failure is as follows:

$$y = \mathbb{E}_{\tilde{X}}[\mathsf{MRL}_X(\tilde{X})],$$

where $\mathsf{MRL}_X(t) = \mathbb{E}_X[X - t | X > t]$ and $\tilde{X} =_d X$. We calculate an estimate of y, denoted by $\hat{y}$ as follows:

- Generate two i.i.d. samples, each of size m, denoted by x_i, $\tilde{x}_i$, $i \in [m]$, $m \in \mathbb{Z}^+$ from the distribution $f(\cdot)$:

$$\begin{aligned} x_i &\sim f(\cdot), i \in [m] \\ \tilde{x}_i &\sim f(\cdot), i \in [m] \end{aligned}$$

- Define $\mathcal{I} \subseteq [m]$ as follows:

$$\mathcal{I} = \{i : i \in [m] \text{ and } x_i > \tilde{x}_i\}$$

- Calculate $\hat{y}$ as follows:

$$\begin{aligned} \hat{y} &= \frac{\sum_{i \in \mathcal{I}} (x_i - \tilde{x}_i)}{|\mathcal{I}|} \\ &= \frac{\sum_{i \in [m]} (x_i - \tilde{x}_i) 1(x_i > \tilde{x}_i)}{\sum_{i \in [m]} 1(x_i > \tilde{x}_i)} \end{aligned}$$

Table I.1 $y_{(n)}$ **for the Gamma(a, b) distribution under $a=b$. The mean is equal to 1, and the variance is equal to $\frac{1}{b}$. The sequence $\frac{y_{(i)}}{i}$ is decreasing in i.**

n	Gamma Distribution ($a=b$) Mean = 1, Variance = $\frac{1}{b}$			
	$a=b=1$	$a=b=2$	$a=b=3$	$a=b=4$
1	1.995	1.707	1.555	1.482
2	3.396	2.972	2.773	2.672
3	4.719	4.179	3.963	3.841
4	5.971	5.381	5.080	4.948
5	7.169	6.496	6.219	6.035
6	8.410	7.633	7.334	7.163
7	9.519	8.772	8.418	8.223
8	10.784	9.904	9.526	9.308
9	11.842	11.020	10.614	10.402
10	13.132	12.089	11.694	11.491

Table I.2 $y_{(n)}$ **for the Lognormal(μ, σ) distribution under $\sigma=\sqrt{2(-\mu)}$. The mean is equal to 1, and the variance is equal to $e^{-(1+2\mu)}$. The sequence $\frac{y_{(i)}}{i}$ is decreasing in i.**

n	Lognormal Distribution ($\sigma=\sqrt{2(-\mu)}$) Mean = 1, Variance = $e^{-(1+2\mu)}$			
	$\mu=-1$	$\mu=-2$	$\mu=-3$	$\mu=-4$
1	3.043	4.114	5.338	5.623
2	5.089	7.013	9.852	10.186
3	6.874	9.885	12.658	12.956
4	8.469	12.267	16.585	20.633
5	10.114	14.317	17.526	23.279
6	11.733	16.612	26.598	27.043
7	13.248	19.361	26.297	30.922
8	14.484	21.093	27.416	38.251
9	15.986	23.452	33.037	39.174
10	17.615	24.682	36.029	41.129

Table I.3 $y_{(n)}$ **for the Weibull(**k,λ**) distribution under** $\lambda = \frac{1}{\Gamma(1+\frac{1}{k})}$**, where** $\Gamma(\cdot)$ **denotes the (complete) Gamma function. The mean is equal to 1, and the variance is equal to** $\frac{\Gamma(1+\frac{2}{k})}{(\Gamma(1+\frac{1}{k}))^2} - 1$**. The sequence** $\frac{y_{(i)}}{i}$ **is decreasing in** i**.**

n	Weibull Distribution ($\lambda = \frac{1}{\Gamma(1+\frac{1}{k})}$) Mean = 1, Variance = $\lambda^2\Gamma(1+\frac{2}{k}) - 1$			
	$\lambda = 1$	$\lambda = 2$	$\lambda = 3$	$\lambda = 4$
1	1.977	1.494	1.337	1.251
2	3.397	2.701	2.469	2.357
3	4.743	3.864	3.565	3.440
4	5.948	4.977	4.655	4.503
5	7.140	6.099	5.736	5.564
6	8.375	7.187	6.803	6.607
7	9.625	8.265	7.880	7.661
8	10.671	9.344	8.931	8.709
9	11.817	10.429	9.990	9.750
10	13.027	11.538	11.057	10.777

Appendix J: Calculation of $y_{[n]}$ and $y_{(i)}$ for Some IFR Non-I.I.D. Task Durations

We calculate $y_{(i)}$ and $y_{[n]}$ for the case of independent, non-i.i.d. task durations, as discussed in Section 5. We restrict attention to the case of two-task processes ($n=2$). Recall from Theorem 7 that task durations drawn from independent, non-identical exponential distributions satisfy (26). Below, we analyze the validity of (26) for the following distributions:

(a) Normal

(b) Uniform

(c) Gamma

(d) Lognormal

(e) Weibull

Specifically, we assume that both tasks are drawn from the same family of distributions (from the list above). In each of these settings, we let $\mathbb{E}[X_1]=1$ (w.l.o.g.). We change the parameters of the distributions to check whether (26) holds. The simulation procedure is as shown in Appendix I.1.

J.1. Normal Distribution

Denote the distribution of task i, $i \in [2]$, by the following:

$$X_i \sim \mathcal{N}\left(\mu_i, \sigma_i^2\right).$$

We choose the parameters μ_i, σ_i, $i \in [2]$, as follows:

$$\begin{aligned} \mu_1 &= 1, \mu_2 \in \{0.25, 0.5, 1, 2, 4\} \\ \sigma_i &= \frac{\mu_i}{3}. \end{aligned}$$

Table J.1 $y_{(i)}$ **and** $y_{[n]}$ **for the Normal**(μ_i, σ_i) **distribution for** $n=2$**. The values of the various parameters are as shown above.**

(μ_2, σ_2)	$(0.25, \frac{0.25}{3})$	$(0.5, \frac{0.5}{3})$	$(1, \frac{1}{3})$	$(2, \frac{2}{3})$	$(4, \frac{4}{3})$
$y_{(1)}$	0.373	0.373	0.373	0.373	0.373
$y_{(2)}$	0.094	0.188	0.376	0.751	1.504
$y_{[2]}$	0.387	0.422	0.534	0.841	1.538
Is $y_{(1)} + y_{(2)} > y_{[2]}$?	True	True	True	True	True

J.2. Uniform Distribution

Denote the distribution of task i, $i \in [2]$ by the following:

$$X_i \sim U[a_i, b_i]$$

Let μ_i and σ_i denote the mean and s.d. of X_i. We have:

$$\mu_i = \frac{b_i - a_i}{2} = \sqrt{3}\sigma_i.$$

We choose the parameters a_i, b_i, $i \in [2]$ as follows:

$$\begin{aligned} a_i &= 0 \\ b_1 &= 1, b_2 \in \{0.5, 1, 2, 4, 8\} \end{aligned}$$

Table J.2 $y_{(i)}$ **and** $y_{[n]}$ **for the Uniform**(a_i, b_i) **distribution for** $n = 2$**. The values of the various parameters are as shown above.**

b_2	0.5	1	2	4	8
$y_{(1)}$	0.665	0.665	0.665	0.665	0.665
$y_{(2)}$	0.167	0.335	0.666	1.334	2.664
$y_{[2]}$	0.688	0.743	0.935	1.483	2.726
Is $y_{(1)} + y_{(2)} > y_{[2]}$?	True	True	True	True	True

J.3. Gamma Distribution

Denote the distribution of task i, $i \in [2]$ by the following:

$$X_i \sim \mathsf{Gamma}(\alpha_i, \beta_i)$$

where α_i (resp., β_i) denotes the shape (resp., rate) parameter. Let μ_i and σ_i denote the mean and s.d. of X_i. We have:

$$\mu_i = \frac{\alpha_i}{\beta_i} \text{ and } \sigma_i = \frac{\sqrt{\alpha_i}}{\beta_i}.$$

We choose the parameters α_i, β_i as follows:

$$\begin{aligned} \alpha_1 &= \beta_i = 1 \\ \alpha_2 &\in \{0.25, 0.5, 1, 2, 4\} \end{aligned}$$

Table J.3 $y_{(i)}$ **and** $y_{[n]}$ **for the Gamma**(α_i, β_i) **distribution for** $n = 2$**. The values of the various parameters are as shown above.**

α_2	0.25	0.5	1	2	4
$y_{(1)}$	1	1	1	1	1
$y_{(2)}$	0.377	0.631	1	1.490	2.181
$y_{[2]}$	1.135	1.274	1.482	1.873	2.463
Is $y_{(1)} + y_{(2)} > y_{[2]}$?	True	True	True	True	True

J.4. Lognormal Distribution

Denote the distribution of task i, $i \in [2]$ by the following:

$$X_i \sim \mathsf{Lognormal}(\mu_i, \sigma_i)$$

where α_i (resp., β_i) denotes the shape (resp., rate) parameter. Let μ_i and σ_i denote the mean and s.d. of X_i. We have:

$$\mathbb{E}[X_i] = e^{\mu_i + \frac{\sigma_i^2}{2}} \text{ and s.d. of } X_i = \left(e^{\mu_i + \frac{\sigma_i^2}{2}}\right)\sqrt{e^{\sigma_i^2} - 1}.$$

We choose the parameters α_i, β_i as follows:

$$\begin{aligned} \mu_1 &= -\frac{1}{2}, \mu_2 \in \left\{-2, -1, -\frac{1}{2}, 0, 1\right\} \\ \sigma_i &= 1 \end{aligned}$$

Table J.4 $y_{(i)}$ **and** $y_{[n]}$ **for the Lognormal**(μ_i, σ_i) **distribution for** $n = 2$**. The values of the various parameters are as shown above.**

α_2	0.25	0.5	1	2	4
$y_{(1)}$	1.044	1.044	1.044	1.044	1.044
$y_{(2)}$	0.230	0.636	1.037	1.704	4.650
$y_{[2]}$	1.104	1.333	1.648	2.195	4.965
Is $y_{(1)} + y_{(2)} > y_{[2]}$?	True	True	True	True	True

J.5. Weibull Distribution

Denote the distribution of task i, $i \in [2]$ by the following:

$$X_i \sim \mathsf{Weibull}(k_i, \lambda_i)$$

where k_i (resp., λ_i) denotes the shape (resp., scale) parameter. Let μ_i and σ_i denote the mean and s.d. of X_i. We have:

$$\mu_i = \lambda_i \Gamma\left(1+\frac{1}{k_i}\right) \text{ and } \sigma_i = \lambda_i \sqrt{\Gamma\left(1+\frac{2}{k_i}\right) - \left(\Gamma\left(1+\frac{1}{k_i}\right)\right)^2}.$$

We choose the parameters α_i, β_i as follows:

$$k_1 = \lambda_i = 1$$
$$k_2 \in \{0.25, 0.5, 1, 2, 4\}$$

Table J.5 $y_{(i)}$ **and** $y_{[n]}$ **for the Weibull**(k_i, λ_i) **distribution for** $n=2$**. The values of the various parameters are as shown above.**

α_2	0.25	0.5	1	2	4
$y_{(1)}$	1	1	1	1	1
$y_{(2)}$	44.146	3	1	0.517	0.287
$y_{[2]}$	44.3	3.415	1.504	1.154	1.054
Is $y_{(1)} + y_{(2)} > y_{[2]}$?	True	True	True	True	True